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# COMPUTATION OF UNSTEADY FLOWS AT ALL SPEEDS WITH A STAGGERED SCHEME

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**Abstract.** *The method designed for accurate and efficient computation of both compressible and incompressible flows previously published [2] and presented at ECCOMAS 1996 [3] for steady flows is applied to unsteady flows. Since in our method the transient behaviour is not falsified, temporal accuracy is obtained in a simple manner, without introducing a pseudo-time variable and dual time stepping as required in many other methods designed to tackle the same problem. We will show that with our staggered scheme unsteady fully compressible flows can be computed as accurately as with colocated compressible schemes. In addition, the Mach number can be prescribed to be arbitrarily small, including zero.*

## 1 INTRODUCTION

When the Mach number in a flow is uniformly low, say below 0.2, solution can take place with equations and numerical methods assuming incompressibility. When the Mach number is higher, the compressible flow equations need to be employed, and numerical methods different from those for the incompressible case are used. This leaves us with the question: what to do when both low and high Mach numbers occur simultaneously in a flow? What is needed is a method with Mach-uniform accuracy and efficiency, both as the Mach number $M \downarrow 0$ and when $M = \mathcal{O}(1)$. Straightforward use of standard compressible methods gives severe convergence problems or even breakdown in the presence of regions with very low Mach number. Therefore efforts have been made to develop special methods for such flows.

For low speed variable density flows asymptotic methods based on series expansion in the Mach number have been developed, a.o. in [7]. Such methods can only be used when the Mach number is small enough (less than 0.3 say). Another approach is to improve the low Mach number behaviour of compressible methods. Because large investments have been made in codes for compressible flows, much work has been done in this direction. Extension to low Mach numbers can be done by preconditioning [11]. This measure usually falsifies the time dependence, making time-accurate unsteady computations awkward or inefficient. Also, usually very small Mach numbers (less than 0.05, say) cannot be handled, or only at increasing expense, and the limit $M \downarrow 0$ is frequently singular.

Alternatively, one may extend an incompressible method to the compressible case. Obviously, this gives the best prospects for handling the limit $M \downarrow 0$, assuming that in this limit a well-proven incompressible scheme is recovered. With a nonstaggered scheme this has been done by Demirdžić et al. [4]. A staggered grid was first used to compute compressible flows by Harlow and Amsden [5], generalizing the MAC scheme of Harlow and Welch [6] to the compressible case, in orthogonal coordinates. For incompressible flows staggered grids are attractive, because no artificial measures need to be taken to avoid spurious pressure oscillations, and the physical boundary conditions suffice. Furthermore, as some of the test cases to be described show, accuracy and efficiency of a staggered scheme turn out to compare quite well with standard schemes in the fully compressible case.

Here we will generalize the incompressible staggered scheme described by Wesseling et al. [14] to the instationary compressible case. This results in a computing method with the following properties: accuracy at low Mach numbers, efficiency at low Mach numbers, applicability to fully compressible time-dependent flow, and applicability to time-dependent incompressible flow. Although other workers have realised some of these issues, we believe combining all of them simultaneously in one method is new.

## 2 DIMENSIONLESS EQUATIONS

The Euler equations are considered. Pressure is used as primitive variable instead of density, in order to handle the limit $M \downarrow 0$. For brevity the equations are presented in Cartesian tensor notation, although in fact they are solved in general coordinates. The dimensional governing equations are:

$$\left(\frac{\partial \rho}{\partial p}\right)_h \frac{\partial p}{\partial t} + \left(\frac{\partial \rho}{\partial h}\right)_p \frac{\partial h}{\partial t} + (\rho u^\alpha)_{,\alpha} = 0, \tag{1}$$

$$\frac{\partial \rho u^\alpha}{\partial t} + \left(\rho u^\alpha u^\beta\right)_{,\beta} = -p_{,\alpha}, \tag{2}$$

$$\frac{\partial h}{\partial t} + u^\alpha h_{,\alpha} = -\left(\gamma - 1\right) h\, u^\alpha_{,\alpha}, \tag{3}$$

where $u^\alpha = u_\alpha$ are the Cartesian velocity components, $\rho$ is the density, $p$ is the pressure, $h$ is the enthalpy, and $\gamma$ is the specific heat ratio. The equation of state for an ideal gas completes the system of equations: $\rho = \frac{\gamma}{\gamma-1}\frac{p}{h}$. Although a nonconservative form for the energy equation is used, the numerical scheme to be used turns out to converge to genuine weak solutions. The above nonconservative form is merely used for greater efficiency in the pressure correction time stepping scheme to be described, and could be replaced by the conservative form.

A special non-dimensionalisation is used such that all dependent variables and coefficients in the governing equations remain finite for $M \geq 0$. This makes the limit $M \downarrow 0$ regular, and avoids problems with rounding errors in discrete schemes. The crucial scaling is the one for the pressure. Reference values are chosen to be: $L, w_\infty, \rho_0$, and $h_0$, where $L$ is a typical length scale of the domain, $w_\infty$ is a measure of the inflow velocity and $\rho_0, h_0$ are the corresponding stagnation conditions:

$$\begin{aligned} h_0 &= \left(1 + \tfrac{\gamma-1}{2} M_\infty^2\right) h_\infty^*, \qquad M_\infty = w_\infty^* / a_\infty^*, \\ \rho_0 &= \left(1 + \tfrac{\gamma-1}{2} M_\infty^2\right)^{\frac{1}{\gamma-1}} \rho_\infty^*, \quad a_\infty^* = \sqrt{(\gamma-1)\, h_\infty^*}, \end{aligned} \tag{4}$$

where the subscript $\infty$ refers to the prescribed inflow velocity, and dimensional quantities are denoted by an asterisk. The dimensionless pressure is

$$p = \frac{p^* - p^*_{out}}{\rho_0 {w_\infty}^2}, \tag{5}$$

where $p^*_{out}$ is the outlet pressure, which is to be prescribed. The nondimensionalisation leaves the equations of motion invariant, but the equation of state becomes:

$$\rho = \rho(p,h) = \frac{\gamma M_\infty^2}{1 + \frac{\gamma-1}{2} M_\infty^2} \frac{p}{h} + \left[ p_v \left( \left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{-\gamma}{\gamma-1}} - 1 \right) + 1 \right] \frac{1}{h}, \tag{6}$$

where $p_v$ is defined by: $p_v = \frac{p^*_{out} - p^*_0}{p^*_\infty - p^*_0}$. Clearly, the limit $M_\infty \downarrow 0$ of the scaled equations is regular. The dimensionless equation of state shows that $\rho$ becomes independent of $p$ as $M_\infty \downarrow 0$, which is precisely what we want, because this eliminates acoustic modes. Furthermore, in this limit the variation of $p$ remains $\mathcal{O}(1)$. This dimensionless formulation includes the incompressible case, which is obtained by putting $M_\infty = 0$.

## 3 SOLUTION STRATEGY

The system of dimensionless equations 1-3 is discretised on a staggered grid using a finite volume technique. The resulting equations are discretised implicitly in time and solved using the pressure correction method, originally developed for incompressible flows.

### 3.1 Discretisation on a staggered grid

The compressible Euler equations will be discretised in space in boundary-fitted coordinates using a finite volume scheme on a staggered grid, for the reasons given in Section 1. Figure 1 shows part of the staggered computational grid, the control volumes and the numbering system. In the case of orthogonal coordinates the scheme is similar to that of Harlow and Amsden [5], except for the pressure scaling (5); in [5] no special scaling is used, hence the limit $M_\infty \downarrow 0$ is singular. The scheme is designed such that as $M_\infty \downarrow 0$ the classical incompressible staggered grid method of Harlow and Welch [6] is recovered (in the Cartesian case). This may be expected to give Mach-independent accuracy and efficiency for small and medium Mach numbers. The discretisation in general coordinates introduces mass flux components $\rho V^\alpha$, $V^\alpha = \sqrt{g} U^\alpha$, $g = \det(g^{\alpha\beta})$. The invariant discretisation in general coordinates is described in detail in [14].

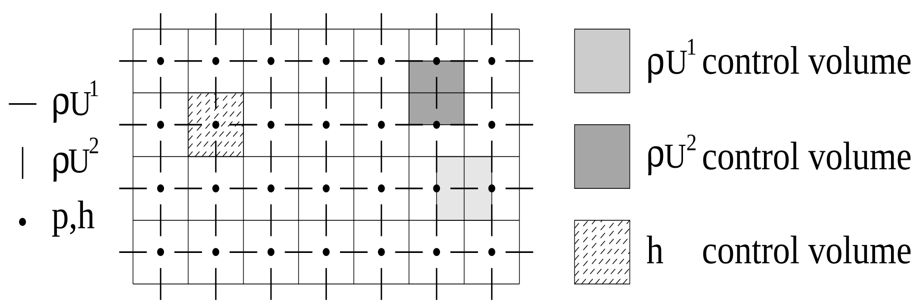


Figure 1: Grid in computational domain $G$

### 3.2 Pressure correction method for compressible flows

In the incompressible case there is no time-derivative for the pressure. This is also the case in the present Mach-uniform formulation, as it should be; for $M_\infty = 0$ the factor $\frac{\partial \rho}{\partial p}$ in equation (1) is zero, according to the dimensionless equation of state. The standard workhorse to do time-stepping in the incompressible case is the pressure correction method, already introduced by Harlow and Welch [6]. This method will also be employed here, for all Mach numbers, in order to achieve Mach-uniform accuracy and efficiency.

The continuity and momentum equation are discretised in time with the first order implicit Euler method. The discretisation of the energy equation is implicit in $h$, but explicit in the other unknowns. For the linearisation of the convection term the Picard method will be used. For the pressure correction equation, first, a prediction for the momentum field $(\rho u^\alpha)^* = m^{\alpha *}$ is computed from

$$\frac{(m^\alpha)^* - (m^\alpha)^n}{\delta t} + \left((m^\alpha)^* \, (U^\beta)^n\right)_{,\beta} = -\left(g^{\alpha\beta} \, p^n\right)_{,\beta}. \tag{7}$$

Next, a pressure correction $\delta p = p^{n+1} - p^n$ is computed. To find the correction equation, first (7) is subtracted from the momentum equation discretised at time level $n+1$, neglecting the difference in the convection terms:

$$\frac{(m^\alpha)^{n+1} - (m^\alpha)^*}{\delta t} = -\left(g^{\alpha\beta} \, \delta p\right)_{,\beta}. \tag{8}$$

It has been shown for the incompressible case that neglecting the difference in the convection term does not adversely effect the temporal accuracy. Next, the discrete divergence of the discretised version of (8) is taken, and the resulting expression for $(m^\alpha_{,\alpha})^{n+1}$ is substituted into the continuity equation, which results in

$$\left(\frac{\partial \rho}{\partial p}\right)_h^n \frac{\delta p}{\delta t} - \delta t \left(g^{\alpha\beta} \, \delta p\right)_{,\alpha\beta} = -(m^\alpha)^*_{,\alpha} - \left(\frac{\partial \rho}{\partial h}\right)_p^n \frac{\delta h}{\delta t}. \tag{9}$$

This is called the pressure correction equation. For the computation of $\partial\rho/\partial p$ and $\partial\rho/\partial h$ the non-dimensional equation of state (6) is used. When $\delta h = h^{n+1} - h^n$ is known, the pressure correction $\delta p$ can be computed from (9), whereafter $(m^\alpha)^{n+1}$ can be found from (8). The $\delta h$ can be computed from the semi-implicit discretisation of the energy equation. In practice the Euler equations are first discretised in space before the pressure correction method is applied, so that the equations derived in this section are linear algebraic systems. No boundary conditions for the pressure correction equation are required, since the physical velocity and pressure boundary conditions have already been incorporated in the continuity and momentum equations, from which the pressure correction equation is derived. The resulting systems of equations are solved by a Krylov subspace iterative method for unsymmetric matrices, GMRES, see [13].

## 4 RESULTS

From various steady test cases discussed in [2], [3] we conclude that with our method flows at all speeds can be efficiently and accurately computed. Here, we will discuss applications with time-dependence. As stated before, one of the advantages of our method over preconditioning methods for weakly compressible flows is that the transient behaviour is not falsified, so that time-accuracy is easily realised. In order to show that for fully compressible unsteady flows the present staggered method is as accurate as colocated compressible schemes, results will be compared with results obtained with the Osher method, described in [9]. This is done for an important test case for compressible flows: the Riemann problem.

### 4.1 Sod's shock tube problem

Sod's shock tube problem [10] is a Riemann proble with the following left and right states:

$$u_4 = 0,\ p_4 = 1.0,\ \rho_4 = 1.0, \quad u_1 = 0,\ p_1 = 0.1,\ \rho_1 = 0.125. \tag{10}$$

As can be seen from the Mach number plot the flow remains subsonic. The maximum

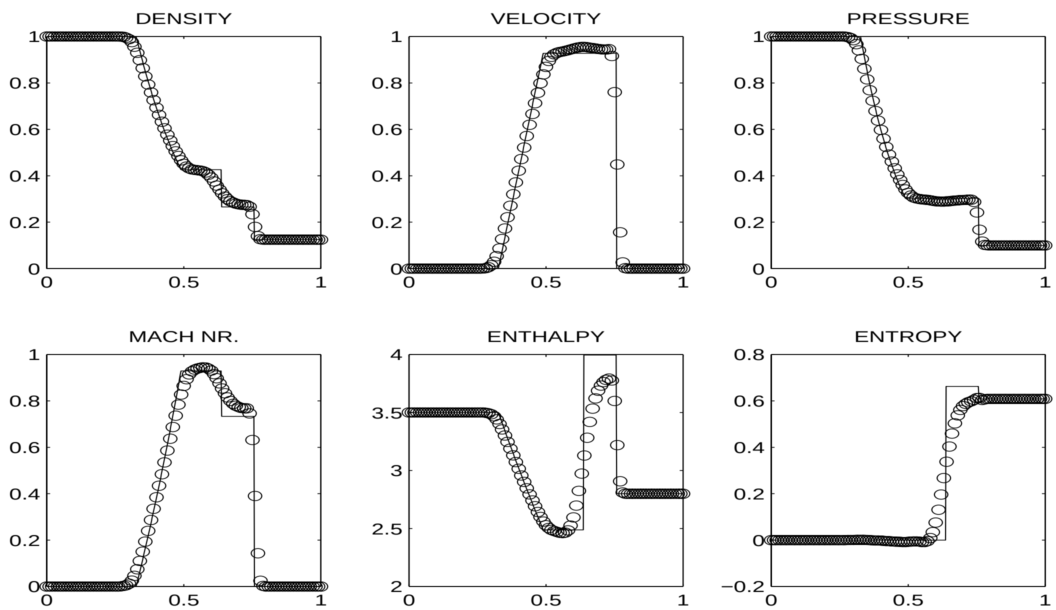


Figure 2: Comparison of first order upwind scheme with exact solution

wave speed in the flow is $\overline{\mu}_m = \max(|u| + a) = 2.2$. Consequently, with $\delta x = 0.01$ the time step restriction for the explicit Osher method is for this test problem $\delta t < 0.0045$. In the computations of Sod's shock tube problem we use $\delta t = 0.004$.

Our solution at $t = 0.15$ is compared with the exact solution at this time in Figure 2. There it can be seen that our scheme converges to the correct weak solution and satisfies the entropy condition, which states that the entropy of fluid particles does not decrease. Of course, over the contact discontinuity the entropy decreases, but since fluid particles do not cross the contact discontinuity, the entropy of the particles does not decrease. Due to the use of first order upwind spatial discretisation, the contact discontinuity is smeared

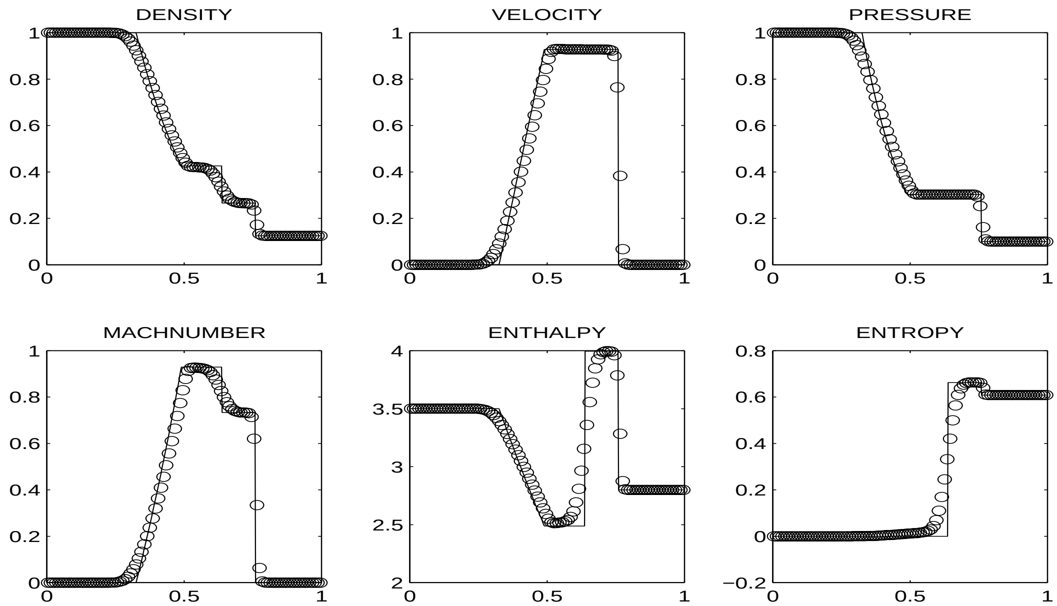


Figure 3: Sod's shock tube problem; comparison of Osher scheme with exact solution at $t = 0.15$

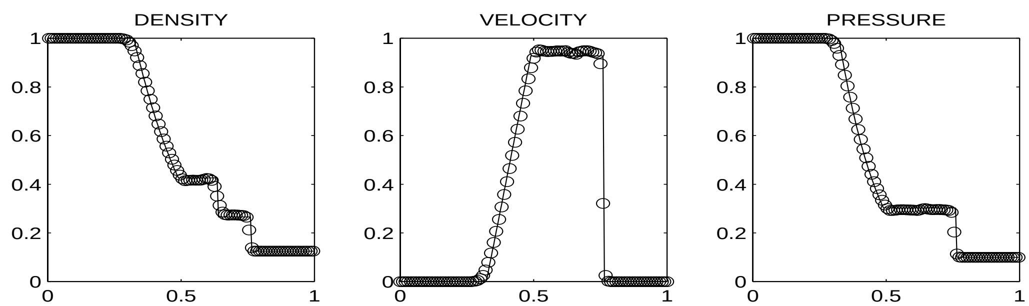


Figure 4: Sod's shock tube problem; comparison of staggered MUSCL scheme with exact solution at $t = 0.15$

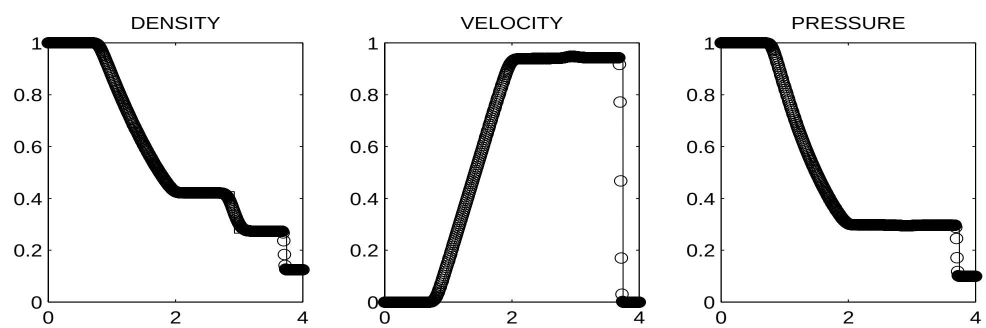


Figure 5: Sod's shock tube problem; comparison of numerical solution with exact solution at $t = 1$.

as can be seen from the enthalpy and entropy plots,and to a lesser extent from the density plot. But the shock resolution is crisp. The numerical shock is seen to move at the correct speed, and the approximation of the expansion wave is smooth.

Solution of Sod's shock tube problem with Osher's approximate Riemann solver [8, 9] is shown in Figure 3. Comparison of Fig. 2 with Fig. 3 shows our method to be about as accurate as the Osher scheme.

The accuracy of the present method can be improved by using a higher order upwind biased scheme. For this we follow the MUSCL approach, see Van Leer [12]. In Figure 4 the solution obtained with the MUSCL scheme is compared to the exact solution. The shock and especially the contact discontinuity have become more crisp. A tiny wiggle is formed in front of the contact discontinuity, which might be due to the non-conservative discretisation of the energy equation. The quality of the results in Figure (4) is about the same as for standard colocated approximate Riemann solvers enhanced with the MUSCL approach (results not shown).

In order to check the computed shock speed a computation over a longer time, i.e. $t = 1$, has been carried out. In this computation the domain was larger: $[0, 4]$, with the separation between the left and right state at $x_0 = 2$. Also in this computation $\delta x = 0.01$, and $\delta x = 0.004$. The result is shown in Fig. 5. There it can be seen that even after computation over a longer time the computed shock position is still right. Although our method is not fully conservative, due to use of the non-conservation form of the energy equation, this does not result in a wrong shock speed. Apparently, use of the conservation form of the continuity and momentum equations is good enough.

### 4.2 Supersonic flow test case

In the previous two test cases the flow remained subsonic. Supersonic flow may bring additional difficulties. Therefore, we will also consider the Mach 3 test case of Arora and Roe [1]. The initial state is specified by

$$u_4 = 0.92,\ p_4 = 10.333,\ \rho_4 = 3.857, \quad u_1 = 3.55,\ p_1 = 1,\ \rho_1 = 1. \tag{11}$$

The flow is characterised by a strong expansion fan. The maximum Mach number is equal to 3. The maximum wave speed in the flow is $\overline{\mu}_m = \max(|u| + a) = 5.0$. Consequently, with $\delta x = 0.01$ the time step restriction for the explicit Osher method is for this test problem $\delta t < 0.002$. In the computations of the test case of Lax we use $\delta t = 0.002$, which did not cause problems.

For the space discretisation of the convection terms in the momentum and energy equation the first order upwind scheme was used. For this test problem the right solution was obtained even without density biased upwinding. The solution obtained with the present method at $t = 0.088$ is compared with the exact solution in Figure 6. There it can be seen that our scheme converges to the correct weak solution.

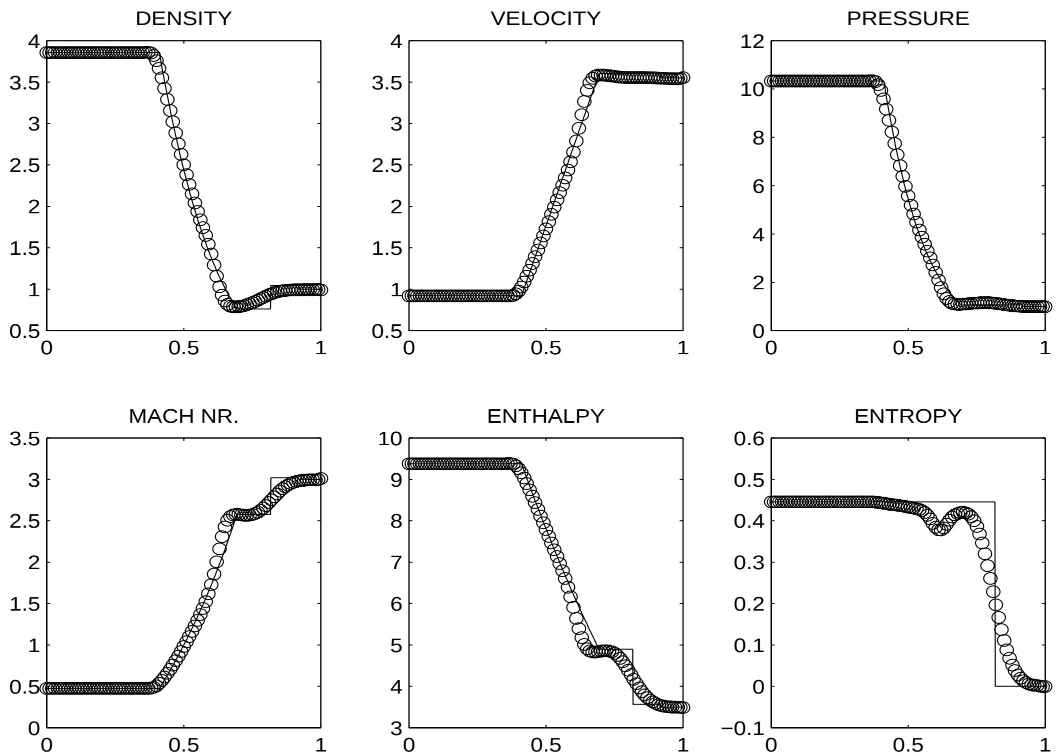


Figure 6: Mach 3 test case; comparison of staggered first-order upwind scheme with exact solution at $t = 0.088$

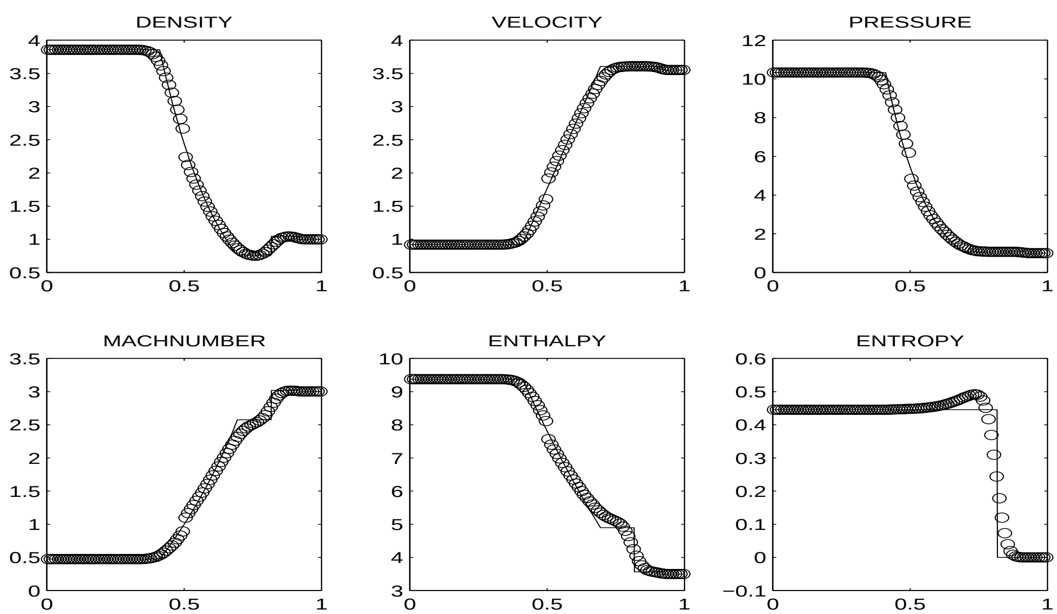


Figure 7: Mach 3 test case; comparison of Osher scheme with exact solution at $t = 0.088$

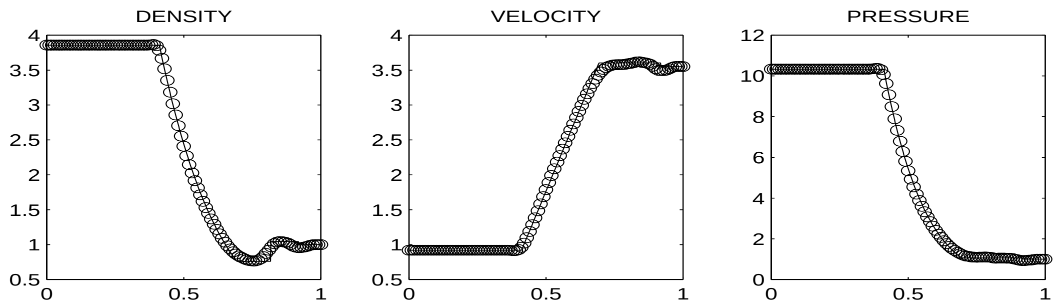


Figure 8: Mach 3 test case; comparison of staggared MUSCL upwind scheme with exact solution at $t = 0.088s$.

Solution of the Mach 3 test case with Osher's approximate Riemann solver [8, 9] is shown in Figure 7. There it can be seen that the Osher scheme suffers from a 'sonic glitch'. That is, near the sonic point the expansion fan shows a discontinuity. But the entropy condition is not violated, as can be seen from the entropy plot, and is in accordance with the theoretical results of [9]. As can be seen from Fig. 6 our scheme does not suffer from this. The contact discontinuity is smeared more with our method.

The accuracy of the numerical solution obtained with the present method is improved by use of the higher order MUSCL scheme. In Figure 8 the solution obtained with the MUSCL scheme is compared to the exact solution. The (weak) shock and especially the contact discontinuity have become more crisp. Again we conclude that the staggered scheme seems to be as accurate as the nonstaggered Osher scheme. Near the sonic point our scheme is even better.

## 5 CONCLUDING REMARKS

Although, as for most methods in current use, a theoretical basis is lacking, on the basis of the experiments described above we have every reason to believe that our scheme approximates genuine weak solutions of the Euler equations that satisfy the entropy condition. Furthermore, our method seems less computing-intensive than colocated methods using approximate Riemann solvers (such as schemes of Osher, Roe and van Leer) or artificial viscosity (Jameson). This is because we use only central and/or upwind differences, which is much cheaper than approximate solutions of Riemann problems to determine numerical fluxes at finite volume boundaries, or using solution dependent second and fourth order artificial dissipation terms.

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