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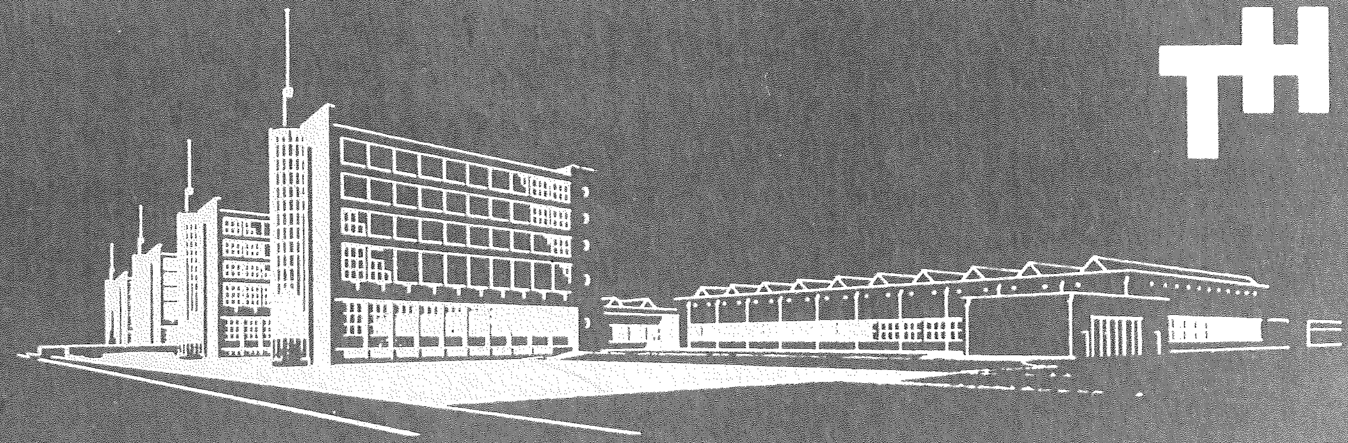

AFDELING DER WERKTUIGBOUWKUNDE

ON THE ROLLING CONTACT OF TWO ELASTIC BODIES IN THE PRESENCE OF DRY FRICTION

by

J. J. Kalker

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DEPARTMENT OF MECHANICAL ENGINEERING
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Errata.

page 23. formula number (2.10) should read (2.20)

page 56. eq. (3.38) $4(A-B)^2$ = etc. should read

$$4(A-B)^2 = \left(\frac{1}{R_1^+} - \frac{1}{R_2^+}\right)^2 + \left(\frac{1}{R_1^-} - \frac{1}{R_2^-}\right)^2 + $$

$$+ 2\left(\frac{1}{R_1^+} - \frac{1}{R_2^+}\right)\left(\frac{1}{R_1^-} - \frac{1}{R_2^-}\right)\cos 2\omega$$

page 66. formulae (4.6) and (4.7)

$(\underline{V}_u^+ - \underline{V}_u^-)$ should read $(\underline{V}_u^- - \underline{V}_u^+)$

page 78. 2nd eq. (4.38)

$\upsilon_y + \phi x$ should read $\upsilon_y + \phi x'$

Table of Contents.

Samenvatting.

Twee zuiver elastische, volkomen gladde omwentelingslichamen worden op elkaar gedrukt, zodat een eindig contactgebied ontstaat. Vervolgens worden zij om hun assen gewenteld zodat zij over elkaar rollen. Indien men een koppel aanbrengt op het ene lichaam en afneemt van het andere, dan blijken de omtreksnelheden van de lichamen niet gelijk te zijn, zelfs indien de overgebrachte kracht kleiner is dan het produkt van wrijvingscoëfficiënt en normaalkracht. Dit verschijnsel wordt de "gemiddelde slip" (Engels: creepage) van de lichamen genoemd. Is er loodrecht op het contactvlak een component van rotatie van de lichamen ten opzichte van elkaar, dan spreekt men van "spin". In deze dissertatie worden de verschijnselen in het contactvlak bestudeerd; in het bijzonder wordt de betrekking gezocht die het verband aangeeft tussen de gemiddelde slip en spin enerzijds en de totale tangentiële kracht, die de lichamen op elkaar uitoefenen, anderzijds.

Na een historische inleiding in Hoofdstuk 1, worden in Hoofdstuk 2 en Hoofdstuk 3 een aantal wiskundige hulpmiddelen besproken, die voor de hier gegeven behandeling van het probleem noodzakelijk zijn. Wat betreft het elastische gedrag worden de omwentelingslichamen door elastische halfruimten benaderd en wij zullen dus de elastische verplaatsingen onderzoeken, die worden teweeggebracht door verdeelde belastingen van verschillende aard, die aangrijpen in een elliptisch gebied gelegen in het overigens spanningsvrije oppervlak van een elastische halfruimte. Dit onderzoek leidt tot het opstellen van een stelsel lineaire vergelijkingen waarmee de verplaatsingen in de belasting kunnen worden uitgedrukt. Dit stelsel is geschikt om de randvoorwaardeproblemen uit de elasticiteitstheorie op te lossen, waartoe sommige contactproblemen aanleiding geven.

In Hoofdstuk 4 keren wij terug tot het oorspronkelijke probleem. De randvoorwaarden worden opgesteld, en het aantal parameters dat het probleem bepaalt, wordt tot vijf teruggebracht. Tevens worden een aantal symmetrie eigenschappen besproken. Hoofdstuk 4 is verder gewijd aan de theorie van twee grensgevallen, t.w. het geval van zeer kleine (infinitesimale) gemiddelde slip en spin, en het geval van zeer grote

gemiddelde slip en spin (volledig doorglijden). De behandelingsmethode van het eerste geval is afkomstig van DE PATER [1], en werd door KALKER [1] toegepast op cirkelvormige contactgebieden. De methode wordt hier toegepast op elliptische contactgebieden, waarbij de theorie van Hoofdstuk 2 wordt gebruikt. Het geval van volledig doorglijden werd reeds behandeld door LUTZ [1,2,3] en WERNITZ [1,2]. Zij losten het probleem op voor het geval dat de gemiddelde slip de richting van een der hoofdassen van de contactellips heeft. De theorie van Hoofdstuk 4 is niet aan deze beperking onderhevig.

In Hoofdstuk 5 wordt een numerieke methode beschreven voor het algemene geval van eindige gemiddelde slip en spin, waarbij al dan niet volledig doorglijden optreedt. Het probleem wordt eerst teruggebracht tot de minimalisatie van een oppervlakte-integraal. Daarna wordt een numerieke methode besproken waarmee de integraal kan worden geminimaliseerd. Er wordt vervolgens uitvoerig ingegaan op het rekenmachineprogramma dat de numerieke methode verwezenlijkt en tenslotte worden de resultaten toegelicht. Er bestaat een redelijke overeenstemming met het experiment.

In Hoofdstuk 6 worden een aantal conclusies getrokken en enige projecten voor nader onderzoek aangeduid.

Summary.

Two purely elastic, perfectly smooth bodies of revolution are pressed together, so that a finite contact area forms. Then they are rotated about their axes, so that they roll over each other. If a couple is applied to one body and taken from the other, the circumferential velocities of the bodies appear to be no longer equal, even in case the force transmitted is smaller than the product of the coefficient of friction and the normal force. This phenomenon was called "creepage" by CARTER [1]. If there is, perpendicular to the contact area, a component of rotation of the bodies with respect to each other, "spin" is said to be present. In this thesis, the phenomena in the contact area are studied and in particular the relationship is sought which connects the creepage and the spin on the one hand, and the total tangential force which the bodies exert upon each other on the other hand.

After a historical introduction in chapter 1, we discuss in chapter 2 and chapter 3 a number of mathematical tools which are needed for our treatment of the problem. As far as the elastic behaviour is concerned, the bodies are approximated by elastic half-spaces. So we investigate the elastic displacements which are due to distributed loads of different types acting in an elliptical area of the surface of an elastic half-space, while outside the elliptical area the surface is free of traction. This investigation leads to the construction of a system of linear equations by means of which the displacements can be expressed in terms of the surface tractions. This system enables us to solve the boundary value problems of the theory of elasticity which correspond to several contact problems. Chapter 3 finishes with an application of this method to a number of well-known contact problems.

In chapter 4 we return to the original problem. The boundary conditions are set up, and the number of parameters defining the problem is reduced to five. Also, a number of symmetry properties is discussed. The remainder of chapter 4 contains the theory of two limiting cases, viz. the case of very small (infinitesimal) creepage and spin, and the case of very large creepage and spin (bodily

sliding). The method of treatment of the former case is due to DE PATER [1], and it was applied by KALKER [1] to circular contact areas. Here, the method is applied to elliptical contact areas, using the theory of chapter 2. The case of bodily sliding has been treated by LUTZ [1,2,3] and WERNITZ [1,2]. They solved the problem for the case that the creepage has the direction of one of the principal axes of the contact ellipse. In chapter 4, this restriction is removed.

In chapter 5 a numerical method is given for the general case of finite creepage and spin, with or without bodily sliding. The problem is first reduced to the minimalisation of a surface integral. Next, a numerical method is discussed by means of which the integral can be minimized. Then we consider the computer programme which realises the numerical method, and finally we discuss the results. These appear to agree reasonably well with the experimental evidence.

In chapter 6 certain conclusions are drawn, and some projects for further research are indicated.

1. Introduction.

Consider two purely elastic, perfectly smooth bodies of revolution, see Fig. 1. They are pressed together with a force N,

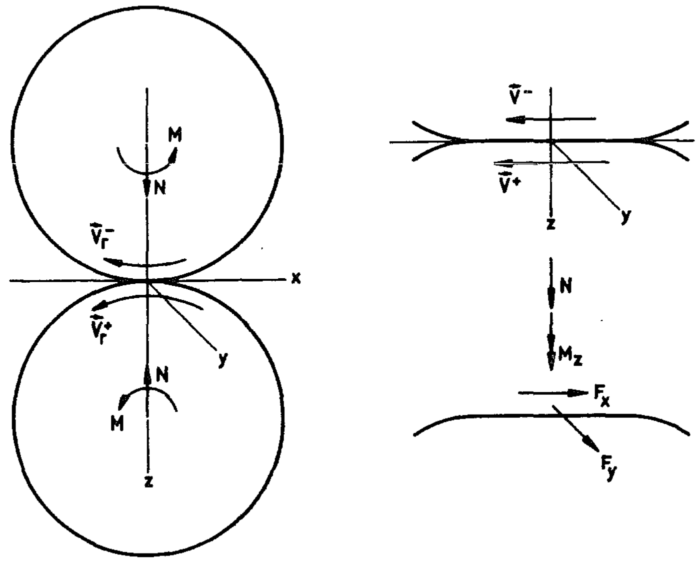

Fig. 1. Two bodies rolling over each other.

as a consequence of which a contact area comes into being along which the bodies touch. According to the theory of HERTZ (see LOVE [1], pg. 193 sqq.), this contact area is an ellipse when the bodies are counterformal. Subsequently, the bodies are rotated about their axes, so that they roll over each other. As a consequence of dry friction, the bodies can exert tangential forces upon each other at the contact area. If a couple is exerted on one body, and taken off from the other, it is found that the circumferential velocities of the bodies are no longer the same, without the occurrence of gross sliding. This difference in the circumferential velocities of the bodies, divided by the rolling velocity, is called the creepage of the bodies. If also the rotations of the bodies about an axis perpendicular to the contact area are different, we speak of spin. The problem is, to investigate what takes place in the contact area, and in particular to find the connection between the two components of creepage (one in the direction of rolling: longitudinal creepage, and one in a direction perpendicular to the rolling direction: lateral creepage) and the spin on the one hand, and the two components of the total tangential force and the moment about an axis perpendicular to the contact area on the other hand.

It is assumed in this work that the law of dry friction (COULOMB's law) with constant coefficient of friction connects the tangential traction at a point of the contact area, and the local velocity of the bodies with respect to each other (the slip), and that a steady state is reached.

1.1. Historical outline.

The problem which we just stated was treated first by CARTER [1] in 1926. He considered the case of two cylinders with parallel axes, in which creepage only occurs in the direction of rolling, and he gave a complete solution of the problem. The tangential stress distribution is found as the difference of two stress distributions which are semicircular when the scale is properly chosen, see fig. 2. One of the stress distributions is acting over the whole contact width, and the other over a part of the contact width, viz. over the region where the local slip is zero: the area of adhesion, or locked area E_h. The area of adhesion is determined by the creepage, here defined as

$$\upsilon_x = \frac{V^- - V^+}{-\frac{1}{2}(V^+ + V^-)}, \qquad (1.1)$$

where V^+ and V^- are the circumferential velocities of the rolling cylinders. The velocity $-\frac{1}{2}(V^+ + V^-)$ which occurs in the denominator of (1.1), is the rolling velocity. The semicircular traction distribution over the whole contact area equals μZ, where Z is the normal pressure distribution and μ is the coefficient of friction. It is a consequence of the semicircular

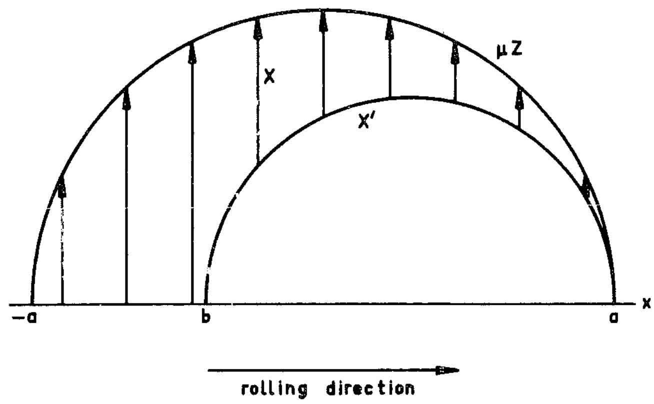

Fig. 2. The tangential stress distribution according to CARTER.

traction distribution over the area of adhesion, that the slip vanishes in the area of adhesion, while the total tangential traction falls below the bound μZ given by the law of friction.

It is seen from Fig. 2 that the adhesion area borders on the leading edge x = a of the contact area. No explanation of this phenomenon was given by CARTER, but it was supplied in 1950 by CAIN [1] in a discussion of a paper by PORITSKY. If the area of adhesion does not border on the leading edge, there would be an area of slip there; but CAIN showed that in that area of slip, the slip does not match the tangential traction as far as the direction is concerned, so that it cannot occur. In the area of slip behind the adhesion area, slip and traction do match in that respect.

The coordinate b of the trailing edge of the contact area is given by

$$\left.\begin{aligned} &b/a = \frac{|\upsilon_x|\rho}{2\mu a} - 1, \quad \text{a: half width of the contact area,} \\ &\frac{1}{\rho} = \frac{1}{4}\left(\frac{1}{R^+} + \frac{1}{R^-}\right), \quad R^+, R^-\text{: radii of cylinders, positive when they are convex.} \end{aligned}\right\} \quad (1.2)$$

It is seen from (1.2) that when the creepage vanishes, then $b/a = -1$, so that the area of adhesion covers the whole contact area, and the tangential traction vanishes. This is called free rolling, in which there is no dissipation by surface friction. There can be dissipation by elastic hysteresis, but such effects are not considered in this work. When the creepage increases, b/a increases, so that the area of adhesion diminishes. When $|\upsilon_x|\rho/\mu a = 4$, b reaches the leading edge of the contact area, and when the creepage increases further, b passes the leading edge. This should be interpreted as follows: no area of adhesion forms at all. The tangential traction equals μZ everywhere, and the slip matches it. This is called gross sliding.

We will give some impression of the magnitude of the creepage in the range we are interested in. When the cylinders have the same radius, then the characteristic length ρ is the diameter of the cylinders. In that case, a representative value of ρ/a is 200, the contact width being dependent on the normal load. A representative value of the coefficient of friction is 0.3. So, when in this example $|\upsilon_x| = 0.003$, the adhesion area covers half of the contact area, and gross sliding sets in when $|\upsilon_x| = 0.006$.

In the region between free rolling and the first onset of gross sliding, the total force F_x exerted on the lower body is given by a parabola which is tangent to the line $F_x = \mu N$, see Fig. 3. In the region of gross sliding, F_x has the maximum value μN.

$$\left.\begin{aligned} F_x &= \frac{1}{16}\,\mu N\left(\frac{\upsilon_x\rho}{\mu a}\right)\left(8 - \frac{|\upsilon_x|\rho}{\mu a}\right), &&\text{if } \frac{|\upsilon_x|\rho}{\mu a} \le 4 \\ &= \mu N, &&\text{if } \frac{|\upsilon_x|\rho}{\mu a} \ge 4 \end{aligned}\right\} \qquad (1.3)$$

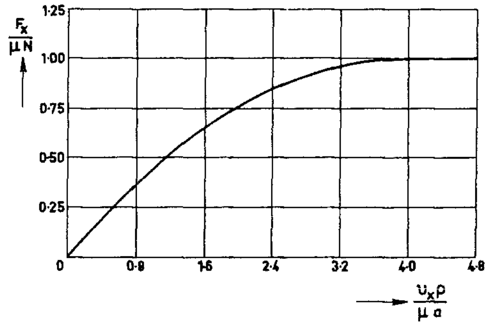

Fig. 3. The total force $F_x/\mu N$ vs. the creepage $\frac{\upsilon_x \rho}{\mu a}$ according to CARTER.

Progress was made by JOHNSON in a number of papers. JOHNSON performed a number of experiments in order to determine the connection between the total tangential force and the torsional moment on the one hand, and creepage and spin on the other hand. In [1] and [5] he also gives a theory of creepage without spin, which is a direct generalisation of CARTER's theory. In this theory, JOHNSON approximates the area of adhesion by an elliptical area which is similar to the contact area, and is similarly oriented. It touches the boundary of the contact area at its foremost point, see Fig. 4. Here also the traction distribution is found in the form of a difference between a semi-ellipsoidal traction distribution acting over the entire contact area, and another, which acts over the adhesion area alone. However, there is a serious flaw in this theory: in the region shown shaded in Fig. 4, the slip and the tangential traction do not match. In fact, if we define the slip as the local velocity of the upper body with respect to the lower, and consider the traction exerted on the lower body, the slip and traction are almost opposite in the shaded area, violating the friction law. In the slip region outside the shaded area, the traction and the slip are almost in the same sense; in fact, they make a small angle, and this is another, smaller, objection

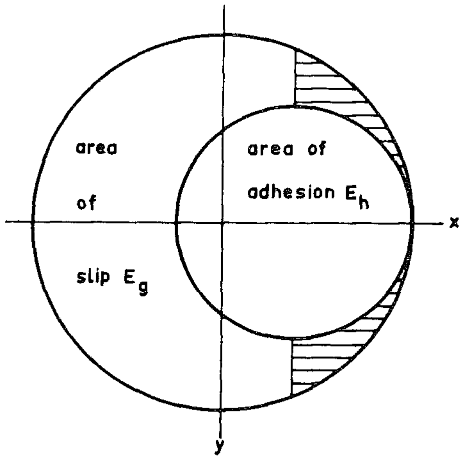

Fig. 4. Areas of adhesion and slip according to JOHNSON.

against the theory. The conclusion we draw from the shaded area of error is, that the area of adhesion is given incorrectly in JOHNSON's theory. If JOHNSON's theoretical results are compared with the experiment, it appears that the theoretical value of the creepage at a certain value of the total force parameter $(F_x, F_y)/\mu N$ is lower than the experimental value. This difference is at most 25%, so that JOHNSON's theory can be used very well as an approximative theory, especially since the values of the coefficient of friction μ differ considerably from one case to another.

Another theory is given by HAINES and OLLERTON [1]. Only creepage in the rolling direction is taken into consideration, and it is assumed that in narrow strips parallel to the rolling direction, CARTER's traction distribution is valid. It then appears that the area of adhesion is given by a lemon shaped area the leading edge of which coincides with the leading edge of the contact area, see Fig. 5. The trailing edge of the adhesion is an arc which, measured along the rolling direction, has a constant distance to the trailing edge of the contact area, in other terms, it is the trailing edge of the contact

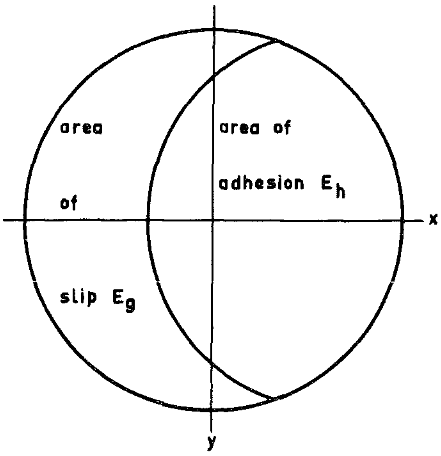

Fig. 5. Areas of adhesion and slip according to HAINES and OLLERTON.

area shifted parallel to itself in the rolling direction. This theory can in principle be used only for contact areas which are slender, with the minor axis in the rolling direction. However, HAINES and OLLERTON have also done photoelastic work from which it appeared that the theoretical form of the area of adhesion was in good agreement with practice, also when the contact area was not slender.

Recently, the theory of HAINES and OLLERTON was generalized by KALKER [2] so, that lateral creepage and, to a limited extent, also spin can be accounted for. In this theory, the elasticity equations are integrated approximately. This approximation is best when the contact ellipse is slender, with the minor semi-axis in the direction of rolling. With this approximate solution of the elasticity equations it is accomplished that 1^o. there is no slip in the adhesion area; 2^o. that the tangential traction in the slip area has the value μZ; but 3^o. there generally remains an angle between traction and slip in the slip area. This angle is small almost everywhere in case of pure creepage and when the spin is small, but deteriorates when the spin increases. When for a slender contact ellipse the total force is compared with the results of ch. 5 of this dissertation, it is found that there is excellent agreement in the case of pure creepage, but in pure spin there are relative errors of up to 20%.

For spin there is a smaller amount of theory than for pure creepage. We just mentioned the theory of KALKER [2]. Aside from that, there are only theories on the two asymptotic cases, viz. very large creepage and spin, and infinitesimal creepage and spin. Experimental work on spin has been done by JOHNSON [2, 3] both on pure spin and on spin in combination with lateral creepage, by LEE and OLLERTON [1], and by POON [1].

The case of very large creepage and spin was treated by LUTZ [1, 2, 3] and WERNITZ [1, 2]. In their theory, they assume that the creepage and spin are so large, that the influence of the elastic deformation on the local slip can be neglected. As a consequence, there is no area of adhesion, and the local slip is completely specified by creepage and spin alone: there is no effect of the tangential traction on the slip. So, the direction of the local slip is known, and hence the direction of the local traction, its magnitude being given by μZ. The total tangential force and the torsional moment follow from integration. LUTZ [2] treated the case of a circular contact area, and WERNITZ [1] the case of an elliptical area. The latter case was treated, however, with a restriction on the components (υ_x, υ_y) of the creepage: either $\upsilon_x = 0$, or $\upsilon_y = 0$. This is the case in friction drives which LUTZ and WERNITZ considered. We will treat the case of very large creepage and spin without this restriction in sec. 4.4 of this dissertation.

The opposite case is the case of infinitesimal creepage and spin. Here it is assumed that the adhesion area covers the entire contact area. For a circular contact area, this case was treated by DE PATER [1] for POISSON's ratio $\sigma = 0$, and by KALKER [1] without this restriction on σ. In sec. 4.3 sqq. of this dissertation, this theory is generalized to elliptical contact areas. Earlier, JOHNSON [2] treated the case of infinitesimal spin for a circular contact area and arbitrary POISSON's ratio. In KALKER [1], a comparison is made between the theories of KALKER [1], JOHNSON [2], and JOHNSON's experiments [2]. There appears to be a fairly large discrepancy between the theories, and KALKER's theory was found to be most in agreement with the experimental results.

In chapter 5 of this dissertation, a numerical theory is developed which can be used for arbitrary creepage and spin. This theory is mainly of academic interest in the case of pure creepage, owing to the fact that the approximative theories are of good quality. In the case of non-vanishing spin, the theory of chapter 5 provides the comparison needed for the safe use of the strip theory; such a comparison is made in KALKER [2]. For values of the spin not covered by the strip theory, the numerical theory of chapter 5 is the only one available. It can also be used to judge, when creepage and spin are large enough so that the theory of LUTZ [1, 2,3] and WERNITZ [1, 2] can be used.

1.2. Two simplifying assumptions. Outline of the thesis.

As far as the theory elasticity is concerned, the lower and the upper body are approximated by half-spaces. In the Cartesian coordinate system (0, x, y, z) which we will adopt, the lower body occupies the half-space $z \geq 0$, and the upper occupies $z \leq 0$. Quantities pertaining to the lower body are distinguished by a superscript $^+$ added to the symbol from the analogous quantity of the upper body which carries a superscript $^-$. The normal pressure is denoted by Z, while we define the tangential tractions (X, Y) as the local tangential (frictional) force per unit area exerted on the lower body by the upper body.

The contact area E and the distribution of normal pressure Z are determined by the boundary conditions of the HERTZ theory; see LOVE [1] pg. 193 sqq.:

$$w(x,y) \equiv w^+(x,y,0)-w^-(x,y,0)= -Ax^2-By^2+\alpha,\quad Z \geq 0 \text{ inside } E, \qquad (1.4a)$$

$$w(x,y) \equiv w^+(x,y,0)-w^-(x,y,0) > -Ax^2-By^2+\alpha,\quad Z=0 \text{ on } z=0, \text{ outside } E, \qquad (1.4b)$$

where $w^{\pm}$ is the displacement component in the z-direction, while $w(x,y)$ is called the displacement difference in the z-direction. A and B are determined by the radii of curvature of the bodies, see (3.38), and α is the penetration of the bodies.

In the first place, we will assume that the tangential traction distribution (X,Y) acting between the bodies does not disturb the displacement difference $w(x,y)$. Such an assumption was already made by MINDLIN [1] in 1949. It was shown by DE PATER [1] pg. 33, that the assumption is completely correct in the case that both bodies have the same elastic constants. A discussion of the error of the approximation when the elastic constants are different will be given in sec. 2.1. The assumption implies that neither the contact area E nor the normal pressure Z are disturbed by the tangential tractions. Consequently, E and Z are given by the HERTZ theory of frictionless contact. According to that theory, which is treated in some detail in sec. 3.221, the contact area E is elliptical in shape, so that we can choose our origin and x and y axes so that

$$E = \{\, x,y,z:\ z = 0,\ (x/a)^2 + (y/b)^2 \le 1 \,\}\ , \qquad (1.5a)$$

while the normal pressure Z is given by

$$Z = \frac{3N}{2\pi ab}\sqrt{1-(x/a)^2 - (y/b)^2} \quad \text{inside } E,$$

$$= 0 \qquad \text{on } z = 0, \text{ outside } E, \qquad (1.5b)$$

N: total normal load.

The local slip at a point is defined as the local velocity of the upper body with respect to the lower body. We ordinarily use the relative slip (s_x,s_y), which is equal to the local slip divided by the rolling velocity. We will show in sec. 4.1 of this dissertation that when steady rolling takes place in the x-direction, the relative slip is given by (4.15):

$$s_x = \upsilon_x - \phi y + \frac{\partial u}{\partial x}\ , \quad s_y = \upsilon_y + \phi x + \frac{\partial v}{\partial x}\ , \qquad (1.6a)$$

with

$$\left.\begin{array}{l}(\upsilon_x,\ \upsilon_y)\text{: the creepage, } \phi\text{: the spin,}\\ u = \{u^+(x,y,0)-u^-(x,y,0)\},\ v = \{v^+(x,y,0)-v^-(x,y,0)\}\\ u^\pm,\ v^\pm\text{: } (x,y) \text{ displacement components in lower/upper body.}\end{array}\right\} \quad (1.6b)$$

We will also assume that the normal pressure distribution Z does not disturb the displacement differences (u,v). Such an assumption was made by MINDLIN [1] in 1949. It was shown by DE PATER [1], pg. 33 that this second assumption is completely correct in the case that the bodies have the same elastic constants. A discussion of the error of the approximation when the elastic constants are different will be given in sec. 2.1.

As a consequence of the assumed independence of w on (X,Y), the problem falls apart into a normal problem which completely determines the normal pressure and the contact area, and a tangential problem which uses the results of the normal problem as data. The reason for the assumed independence of (u,v) on Z lies in the fact that the case of equal elastic constants is technically the most important, while the theory becomes somewhat simpler, and the coefficient of friction does not figure as an independent parameter in the calculation.

A method to obtain a better approximation was indicated by JOHNSON [4], pg. 18 sqq. JOHNSON proposes to retain the assumption that w is independent of (X,Y), but to take the dependence of (u,v) on Z into account. The value of this method consists of the fact that the dependence of (u,v) on Z is much more important than the dependence of w on (X,Y), especially when the coefficient of friction μ is small, see sec. 2.1. The advantage over the rigorous theory is, that the normal problem remains the same, and that the tangential problem changes only in that a term is added to the formula for the relative slip, the term being explicitly known, and being independent of the creepage and the spin. This method is not investigated further in this thesis, where we will retain the two assumptions of MINDLIN.

The tangential problem is determined by the following conditions.

$$\left.\begin{array}{l}(X,Y) \text{ and } (u,v) \text{ are connected by the elasticity equations} \\ \text{for the half-space, in which stresses and displacements} \\ \text{vanish at infinity, while } X = Y = 0 \text{ on } z = 0\text{, outside } E;\end{array}\right\} \quad (1.7)$$

$$\left.\begin{array}{l}(X,Y) = \mu Z(w_x,w_y),\; w_x = s_x/s,\; w_y = s_y/s,\; s = \sqrt{s_x^2 + s_y^2} \\ \qquad \text{in the area of slip } E_g;\end{array}\right\} \quad (1.8a)$$

$$s_x = s_y = 0,\; |(X,Y)| \leq \mu Z \text{ in the area of adhesion } E_h. \quad (1.8b)$$

We see from (1.7) and (1.8) that the tangential problem naturally falls into two parts. In the first part, we must study the effect of the traction distribution (X,Y) on the displacement differences (u,v), in order to get the connection between the traction and the slip. We solve this problem by giving this connection for certain standard traction distributions which form a complete system. In the second part we superimpose the standard tractions so as to fit (approximately) the boundary conditions (1.8). It should be noted that the division of the contact area into areas of slip and adhesion is not known beforehand, but must result from the calculations.

In chapters 2 and 3 of the thesis, we attack the first subproblem, viz. finding a complete set of tractions with their corresponding displacements differences. Apart from the tangential problem in which (X,Y) are given and Z is unimportant as we have here, we also treat the normal problem where (X,Y) are unimportant, Z is arbitrarily prescribed. This is done because it widens the scope of chapters 2 and 3, while it is done without much trouble, since a normal problem is equivalent to a tangential problem in which POISSON's ratio σ vanishes.

In chapter 2, we give the theory of tractions of the form

$$(X,Y,Z) = \{1-(x/a)^2-(y/b)^2\}^{-\frac{1}{2}} \sum_{p+q=0}^{M} (d_{pq},e_{pq},f_{pq})x^p y^q. \quad (1.9)$$

It is shown in 2.2 that to the tractions (1.9) surface displacement differences belong

$$(u,v,w) = \sum_{m+n=0}^{M} (a_{mn},b_{mn},c_{mn})x^m y^n \text{ if } (x,y) \text{ in } E. \qquad (1.10)$$

The remainder of chapter 2 is devoted to the connection between the (a_{mn},b_{mn},c_{mn}) and the (d_{pq},e_{pq},f_{pq}). This connection is given in the form of a square set of linear equations, which we call the <u>load-displacement equations</u>. They express (a_{mn},b_{mn},c_{mn}) explicitly in (d_{pq},e_{pq},f_{pq}).

In chapter 3, we treat special cases of the load displacement equations. In 3.1, we consider the special case that (X,Y,Z) vanish at the edge of the contact area, and have the form

$$(X,Y,Z)=\{1-(x/a)^2-(y/b)^2\}^{+\frac{1}{2}} \sum_{p+q=0}^{M-2} (d_{pq},e_{pq},f_{pq})x^p y^q \qquad (1.11)$$

Again, (u,v,w) are given by (1.10). The coefficients of the load-displacement equations appear to undergo only minor changes. In 3.2, we treat a number of examples, viz. a rigid, flat die of elliptic circumference pressed into a half-space (3.211), the problem of CATTANEO and MINDLIN without slip (3.212), the problem of HERTZ, fairly detailed because it is used later on (3.221), and finally the problem of CATTANEO and MINDLIN with slip, without twist (3.222).

In chapters 4 and 5, we attack the second subproblem, viz. the fitting of the boundary conditions (1.8), by means of the theory of chapters 2 and 3. In 4.1, the boundary conditions are derived; this is followed by considerations of symmetry in 4.2. The remainder of chapter 4 is devoted to the two limiting cases, viz. infinitesimal creepage and spin (sec. 4.3), and very large creepage and spin (sec. 4.4). The case of infinitesimal creepage and spin, which was treated before by DE PATER [1] and KALKER [1] is extended to the case of an elliptical contact area. Tractions of the form (1.9) are used. The case of very large creepage and spin, which was treated by WERNITZ for elliptical areas only when $\upsilon_x=0$ or $\upsilon_y=0$, is here extended to the case of arbitrary creepage. The method of LUTZ and WERNITZ is retained, and the theory of chapters 2 and 3 is not used.

In chapter 5 we treat the case of arbitrary creepage and spin. The procedure is, to write the boundary conditions (1.8) in the form

$$\left.\begin{aligned} I \equiv \iint_E \{1-(x/a)^2-(y/b)^2\}\{(X'-w_x)^2+(Y'-w_y)^2\}\{s_x{}^2+s_y{}^2\}dxdy=0 \\ |(X',Y')| \leq 1 , \end{aligned}\right\} \quad (1.12a)$$

$$\left.\begin{aligned} \text{with } (X,Y) = \mu Z(X',Y') = \frac{3\mu N}{2\pi ab}\{1-(x/a)^2-(y/b)^2\}^{+\frac{1}{2}}(X',Y'), \\ (X',Y') = \sum_{p+q=0}^{M} (d_{pq},e_{pq})x^p y^q, \; M \to \infty . \end{aligned}\right\} \quad (1.12b)$$

It should be observed that (1.12a) can only be satisfied if at every point of the contact area at least one of the factors of the integrand vanishes. The first factor does not vanish except on the edge of the contact area; if the second factor vanishes, (1.8a) is satisfied, and the point belongs to the area of slip; if the second factor vanishes, then (1.8b) is satisfied, and the point belongs to the area of adhesion. The inequality $|(X',Y')| \leq 1$ ensures that the maximum μZ of the tangential traction is not exceeded. We see from (1.12b) that the tractions (1.11) of sec. 3.1 are used. This is done with the purpose to enter a rudiment of the inequality into the integral. In practice, we take $M = 3$ in (1.12b), and minimize I with respect to (d_{pq},e_{pq}), since the positive definite integral I vanishes only for infinite M. The inequality of (1.12a) will be verified afterwards. It is seen that in this method the difference between the locked areas E_h and the slip areas E_g disappears from the problem. The domain of slip can, however, be identified with the area in which $\{(X'-w_x)^2+(Y'-w_y)^2\} \ll (s_x{}^2+s_y{}^2)$, and the domain of adhesion E_h is that in which $\{(X'-w_x)^2+(Y'-w_y)^2\} \gg (s_x{}^2+s_y{}^2)$. This distinction is especially sharp in the case of pure creepage. The calculations were performed for a large number of parameter combinations υ_x, υ_y, ϕ, and a/b (= ratio of the axes of the contact ellipse). In 5.1 sqq, the theory is discussed; in 5.2 sqq, we present some considerations on the computer programme with special emphasis on the optimalisation of the programme and the verification of the inequality, and in 5.3 sqq. we devote our attention to the numerical results.

The dissertation finishes with a conclusion in which the results achieved are summarized, and in which we make some remarks regarding further research.

2. Two elastic half-spaces under normal and shearing loads acting in an elliptical contact area.

In the present chapter, we will consider the stresses and displacement differences that arise when two half-spaces are in contact. Throughout the chapter we assume that contact takes place along an elliptical contact area E.

We introduce a cartesian coordinate system (0,x,y,z), the origin of which lies in the centre of the contact ellipse. The directions of x and y are the axes of the ellipse, and the axis of z is directed along the inner normal of the lower half-space. We denote the surface tractions by (X,Y,Z), the elastic displacement in the lower half-space $z \geq 0$ by (u^+,v^+,w^+), and the elastic displacement in the upper half-space $z \leq 0$ by (u^-,v^-,w^-).

We saw in the previous chapter that as a consequence of our assumptions we could decompose the problem into two partial problems, viz. the normal and the tangential problem.

The normal problem has to be solved first, and it is equivalent to a contact problem without friction. Its boundary conditions are formulated in terms of Z and the displacement difference $w(x,y)=w^+(x,y,0)-w^-(x,y,0)$, and the most important condition is that w(x,y) takes on a prescribed value in E. We can schematize the elasticity part of the problem by solving the following

Normal problem: The shear tractions (X,Y) vanish identically on the whole of the boundary z = 0, and the normal traction Z vanishes outside the elliptical area E. The surface displacement difference w(x,y) is given at E as a polynomial of degree M in x and y:

$$w(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{M-m} c_{mn} x^m y^n \text{ inside E.} \tag{2.1}$$

Find the normal traction Z acting at the area E.

This problem seems to be artificial. The reason why we restrict ourselves to polynomial displacement differences is, that for such a displacement we can find the normal traction Z by solving a finite set of linear equations. Moreover, we observe that the polynomials are complete in the sense that they can approximate any continuous function as well as one likes. Finally, in several problems, e.g. the problem of HERTZ (sec. 3.221), and the problem of a flat rigid die of elliptical circumference that is pressed into a half-space (sec. 3.211), the displacement difference w is actually a polynomial.

Making use of the results of the normal problem, we proceed to solve the tangential problem. From a point of view of elasticity alone, this problem is equivalent to a problem in which there is no normal load at the boundary, as a consequence of the second assumption of MINDLIN, see sec. 1.2. The most important boundary condition in the area of adhesion is the (almost complete) prescription of $(u(x,y),\ v(x,y)) = (u^{+}(x,y,0)-u^{-}(x,y,0),\ v^{+}(x,y,0)+ -v^{-}(x,y,0))$ in it. Hence it is desirable to solve the following

<u>Tangential Problem:</u> The normal traction Z vanishes identically on the entire boundary z = 0, and the tangential surface traction (X,Y) vanishes outside the elliptical area E. The displacement differences (u(x,y), v(x,y)) are given in E as polynomials of degree M in x and y:

$$(u(x,y),\ v(x,y)) = \sum_{m=0}^{M} \sum_{n=0}^{M-m} (a_{mn}, b_{mn}) x^m y^n \text{ inside E.} \qquad (2.2)$$

Find the tangential traction (X,Y) acting at E.

This problem, too, can be solved explicity, in the same way as the normal problem. As in the normal problem, there is an example in which (u,v) are actually polynomials: it is the problem of CATTANEO [1] and MINDLIN [1], in which two bodies are pressed together and then are shifted or twisted, while slip is assumed to be absent. This problem is treated in sec. 3.212.

We finally observe that both problems reduce to problems of the single half-space, when one of the two elastic half-spaces is assumed to be perfectly rigid.

<u>2.1. Formulation of the problems as integral equations.</u>

The connection of the surface tractions and the displacement of a half-space can be given by an integral representation. In order to find it, we observe that the displacement in the lower half-space due to a concentrated load of magnitude Z acting at the origin in the direction of the positive z-axis is given by LOVE [1], par. 135, pg. 191, as follows:

$$\left.\begin{aligned} u^+ &= \frac{Z}{4\pi\mu}\frac{xz}{r^3} - \frac{Z}{4\pi(\lambda+\mu)}\frac{x}{r(z+r)}, \\ v^+ &= \frac{Z}{4\pi\mu}\frac{yz}{r^3} - \frac{Z}{4\pi(\lambda+\mu)}\frac{y}{r(z+r)}, \\ w^+ &= \frac{Z}{4\pi\mu}\frac{z^2}{r^3} + \frac{Z(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\frac{1}{r}, \quad r = \sqrt{x^2+y^2+z^2} \end{aligned}\right\} \tag{2.3}$$

where λ and μ are LAME's constants, which are connected with the modulus of rigidity G and POISSON's ratio σ by the relations

$$\mu = G, \ \lambda = \frac{2\sigma G}{1-2\sigma}, \ \lambda+\mu = \frac{G}{1-2\sigma}, \ \lambda+2\mu = \frac{2G(1-\sigma)}{1-2\sigma}. \tag{2.4}$$

So, (2.3) becomes

$$\left.\begin{aligned} u^+ &= \frac{Z}{4\pi G^+}\left\{\frac{xz}{r^3} - \frac{(1-2\sigma^+)x}{r(z+r)}\right\}, \\ v^+ &= \frac{Z}{4\pi G^+}\left\{\frac{yz}{r^3} - \frac{(1-2\sigma^+)y}{r(z+r)}\right\}, \\ w^+ &= \frac{Z}{4\pi G^+}\left\{\frac{z^2}{r^3} + \frac{2(1-\sigma^+)}{r}\right\}. \end{aligned}\right\} \tag{2.5}$$

The displacement in the lower body due to a distributed pressure Z(x,y) in the z-direction is then given by superposition:

$$\left.\begin{aligned} u^+(x,y,z) &= \frac{1}{4\pi G^+}\iint_E Z(x',y')\left\{\frac{(x-x')z}{r^3} - \frac{(1-2\sigma^+)(x-x')}{r(z+r)}\right\} dx'dy' \\ v^+(x,y,z) &= \frac{1}{4\pi G^+}\iint_E Z(x',y')\left\{\frac{(y-y')z}{r^3} - \frac{(1-2\sigma^+)(y-y')}{r(z+r)}\right\} dx'dy' \\ w^+(x,y,z) &= \frac{1}{4\pi G^+}\iint_E Z(x',y')\left\{\frac{z^2}{r^3} + \frac{2(1-\sigma^+)}{r}\right\} dx'dy', \\ r &= \sqrt{(x-x')^2+(y-y')^2+z^2}, \ z \geq 0. \end{aligned}\right\} \tag{2.6}$$

We must also have the displacement in the upper body. It is due to the reaction of Z(x,y), and consequently it is given by the same equations, but in a coordinate system (x,y,z'), where z' = -z. To find it in our coordinate system (x,y,z), we must change z to $|z|$, and w^+ to $-w^-$ everywhere. This gives for the displacement in both half-spaces:

$$\left.\begin{aligned}
u^{\mp}(x,y,z) &= \frac{1}{4\pi G^{\mp}} \iint_E Z(x',y') \left\{ \frac{(x-x')|z|}{r^3} - \frac{(1-2\sigma^{\mp})(x-x')}{r(|z|+r)} \right\} dx'dy', \\
v^{\mp}(x,y,z) &= \frac{1}{4\pi G^{\mp}} \iint_E Z(x',y') \left\{ \frac{(y-y')|z|}{r^3} - \frac{(1-2\sigma^{\mp})(y-y')}{r(|z|+r)} \right\} dx'dy', \\
w^{\mp}(x,y,z) &= \frac{\mp 1}{4\pi G^{\mp}} \iint_E Z(x',y') \left\{ \frac{z^2}{r^3} + \frac{2(1-\sigma^{\mp})}{r} \right\} dx'dy', \\
r &= \sqrt{(x-x')^2+(y-y')^2+z^2}, \text{ upper and lower sign as } z < 0,\ z > 0.
\end{aligned}\right\} \tag{2.7}$$

From this we see that in case G and σ are the same in both bodies (elastic symmetry),

$$\left.\begin{aligned}
u^+(x,y,z) &= u^-(x,y,-z), \\
v^+(x,y,z) &= v^-(x,y,-z), \\
w^+(x,y,z) &= -w^-(x,y,-z),
\end{aligned}\right\} \text{ if } X = Y = 0 \tag{2.8}$$

a result due to DE PATER [1], pg. 33.

The displacement differences, which are prescribed in the normal and tangential problems, are:

$$\left.\begin{aligned}
u(x,y) &= \{u^+(x,y,0)-u^-(x,y,0)\} = \\
&= \frac{1}{4\pi} \left\{ \frac{1-2\sigma^+}{G^+} - \frac{1-2\sigma^-}{G^-} \right\} \iint_E Z(x',y') \frac{x'-x}{R^2} dx'dy', \\
v(x,y) &= \{v^+(x,y,0)-v^-(x,y,0)\} = \\
&= \frac{1}{4\pi} \left\{ \frac{1-2\sigma^+}{G^+} - \frac{1-2\sigma^-}{G^-} \right\} \iint_E Z(x',y') \frac{y'-y}{R^2} dx'dy', \\
w(x,y) &= \{w^+(x,y,0)-w^-(x,y,0)\} = \\
&= \frac{1}{2\pi} \left\{ \frac{1-\sigma^+}{G^+} + \frac{1-\sigma^-}{G^-} \right\} \iint_E Z(x',y') \frac{dx'dy'}{R}, \\
&X = Y = 0,\ R = \sqrt{(x-x')^2+(y-y')^2}.
\end{aligned}\right\} \tag{2.9}$$

We combine σ^+, σ^- and G^+, G^- in the following manner:

$$\frac{1}{G} = \frac{1}{2}\left(\frac{1}{G^+} + \frac{1}{G^-}\right),\ \frac{\sigma}{G} = \frac{1}{2}\left(\frac{\sigma^+}{G^+} + \frac{\sigma^-}{G^-}\right),\ \kappa = \frac{1}{4} G \left(\frac{1-2\sigma^+}{G^+} - \frac{1-2\sigma^-}{G^-}\right). \tag{2.10}$$

It is easy to see that G lies between G^+ and G^-, and that σ lies between σ^+ and σ^-; in the case of elastic symmetry,

$$G = G^+ = G^-, \ \sigma = \sigma^+ = \sigma^-, \ \kappa = 0. \qquad (2.10a)$$

The constant κ vanishes in case of elastic symmetry, and also when both bodies are incompressible. Its maximum is 0.5, but in practice it is mostly small, e.g. 0.03 for steel on brass, and 0.09 for steel on aluminium. In terms of the constants of (2.10), the displacement differences become

$$\left.\begin{aligned} u(x,y) &= -\frac{\kappa}{\pi G} \iint_E Z(x',y') \frac{x-x'}{R^2}\, dx'dy', \qquad &(a)\\ v(x,y) &= -\frac{\kappa}{\pi G} \iint_E Z(x',y') \frac{y-y'}{R^2}\, dx'dy', \qquad &(b)\\ w(x,y) &= \frac{1-\sigma}{\pi G} \iint_E Z(x',y') \frac{dx'dy'}{R}\, . \qquad &(c) \end{aligned}\right\} \qquad (2.11)$$

If w is prescribed in the contact area E, (2.11c) is an integral equation for the unknown normal pressure Z(x,y).

The procedure for the tangential problem is very nearly the same. We start with the displacement in the lower body due to a concentrated load of magnitude X acting at the origin in the direction of the positive x-axis, see LOVE [1], par. 166, pg. 243,

$$\left.\begin{aligned} u^+ &= \frac{X}{4\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r} + \frac{x^2}{r^3} \right) - \frac{X}{2\pi(\lambda+\mu)}\frac{1}{r} + \frac{X}{4\pi(\lambda+\mu)}\left(\frac{1}{z+r} - \frac{x^2}{r(z+r)^2} \right),\\ v^+ &= \frac{X}{4\pi\mu}\frac{xy}{r^3} - \frac{X}{4\pi(\lambda+\mu)} \cdot \frac{xy}{r(z+r)^2},\\ w^+ &= \frac{X}{4\pi\mu}\frac{xz}{r^3} + \frac{X}{4\pi(\lambda+\mu)} \cdot \frac{x}{r(z+r)},\\ r &= \sqrt{x^2+y^2+z^2}. \end{aligned}\right\} \qquad (2.12)$$

The effect of a distributed shear stress X(x,y) in the x-direction is found by superposition. The displacement due to a load Y in the y-direction is found from (2.12) by cyclic interchange of x and y, u and v, X and Y. The displacement in the upper half-space is given by (2.12) in a coordinate system (x,y,z'), with z' = -z. However, we must take into account that the shearing traction on the upper body has the opposite sign. So we find the displacement in the coordinate system (x,y,z) by replacing X by -X, Y by -Y, z by $|z|$, w^+ by $-w^-$, and it is for both half-spaces

$$u^{\mp}(x,y,z) =$$
$$= \mp \frac{1}{4\pi G^{\mp}} \iint_E \left[X(x',y')\left\{ \frac{1}{r} + \frac{1-2\sigma^{\mp}}{|z|+r} + \frac{(x-x')^2}{r^3} - \frac{(1-2\sigma^{\mp})(x-x')^2}{r(|z|+r)^2} \right\} + \right.$$
$$\left. + Y(x',y')\left\{ \frac{(x-x')(y-y')}{r^3} - \frac{(1-2\sigma^{\mp})(x-x')(y-y')}{r(|z|+r)^2} \right\}\right] dx'dy',$$

$$v^{\mp}(x,y,z) =$$
$$= \mp \frac{1}{4\pi G^{\mp}} \iint_E \left[X(x',y')\left\{ \frac{(x-x')(y-y')}{r^3} - \frac{(1-2\sigma^{\mp})(x-x')(y-y')}{r(|z|+r)^2} \right\} + \right.$$
$$\left. + Y(x',y')\left\{ \frac{1}{r} + \frac{1-2\sigma^{\mp}}{|z|+r} + \frac{(y-y')^2}{r^3} - \frac{(1-2\sigma^{\mp})(y-y')^2}{r(|z|+r)^2} \right\}\right] dx'dy',$$

$$w^{\mp}(x,y,z) =$$
$$= \frac{1}{4\pi G^{\mp}} \iint_E \left[X(x',y')\left\{ \frac{(x-x')|z|}{r^3} + \frac{(1-2\sigma^{\mp})(x-x')}{r(|z|+r)} \right\} + \right.$$
$$\left. + Y(x',y')\left\{ \frac{(y-y')|z|}{r^3} + \frac{(1-2\sigma^{\mp})(y-y')}{r(|z|+r)} \right\}\right] dx'dy',$$

$$r = \sqrt{(x-x')^2+(y-y')^2+z^2}, \quad Z = 0.$$

Upper sign: upper half-space, lower sign: lower half-space. (2.13)

From this we see that in case G and σ are the same in both bodies (elastic symmetry),

$$\left.\begin{aligned} u^+(x,y,z) &= -u^-(x,y,-z), \\ v^+(x,y,z) &= -v^-(x,y,-z), \\ w^+(x,y,z) &= +w^-(x,y,-z), \end{aligned}\right\} \text{ if } Z = 0, \qquad (2.14)$$

a result due to DE PATER [1], pg. 33.

The displacement differences $u(x,y)$, $v(x,y)$, $w(x,y)$, which are prescribed in the normal and tangential problems, become with the definition (2.10) of G, σ, and κ,

$$u(x,y) =$$
$$= \frac{1}{\pi G} \iint_E \left[X(x',y')\left\{\frac{1-\sigma}{R} + \frac{\sigma(x-x')^2}{R^3}\right\} + Y(x',y') \frac{\sigma(x-x')(y-y')}{R^3}\right] dx'dy',$$
$$= \frac{1}{\pi G} \iint_E \left[X(x',y')\left\{\frac{1}{R} - \sigma \frac{\partial^2 R}{\partial x'^2}\right\} - \sigma\, Y(x',y') \frac{\partial^2 R}{\partial x' \partial y'}\right] dx'dy',$$

(2.15a)

$$
\left.\begin{aligned}
v(x,y) &= \\
&= \frac{1}{\pi G} \iint_E \left[X(x',y')\,\frac{\sigma(x-x')(y-y')}{R^3} + Y(x',y')\left\{\frac{1-\sigma}{R} + \frac{\sigma(y-y')^2}{R^3}\right\}\right]dx'dy' \\
&= \frac{1}{\pi G} \iint \left[-\sigma\, X(x',y')\,\frac{\partial^2 R}{\partial x'\partial y'} + Y(x',y')\left\{\frac{1}{R} - \sigma\,\frac{\partial^2 R}{\partial y'^2}\right\}\right]dx'dy',
\end{aligned}\right\} \qquad (2.15b)
$$

$$
w(x,y) = \frac{\kappa}{\pi G} \iint_E \left[X(x',y')\,\frac{x-x'}{R^2} + Y(x',y')\,\frac{y-y'}{R^2}\right]dx'dy', \qquad (2.15c)
$$

$$
Z = 0, \; R = \sqrt{(x-x')^2+(y-y')^2}. \qquad (2.16)
$$

If $Z = 0$, and u and v are prescribed in the contact area, (2.15a) and (2.15b) are two simultaneous integral equations for the unknown tangential tractions (X,Y).

According to (2.11) and (2.15), we see that a rough estimate of (u,v,w) in the contact area is

$$
\left.\begin{aligned}
&u = O(F_x/Gs) + O(\sigma F_y/Gs) + O(\kappa N/Gs), \\
&v = O(\sigma F_x/Gs) + O(F_y/Gs) + O(\kappa N/Gs), \\
&w = O(\kappa F_x/Gs) + O(\kappa F_y/Gs) + O\big((1-\sigma)N/Gs\big), \\
&F_x,\ F_y,\ N\text{: total force in the } x,y,z\text{-directions}, \\
&s\text{: half diameter of the contact area.}
\end{aligned}\right\} \qquad (2.17)
$$

Throughout the present work we will neglect the influence of the small constant κ. This leads to exact results in the technically important case of elastic symmetry, and also when both bodies are incompressible.

It would seem that our approximation leads to a high precision in the case of w, since F_x and F_y are the most of the order μN (μ: coefficient of friction), so that the influence of X and Y on w is of $O(\mu\kappa N/Gs)$, which seems to be negligible with respect to the influence of Z, which is of $O\big((1-\sigma)N/Gs\big)$. But neglecting the influence of Z on (u,v) can lead to serious errors: this influence can be of $O(\kappa N/Gs)$, while the influence of the tangential traction is of $O(\mu N/Gs)$. Hence we would obtain a good second approximation by taking the influence of Z on (u,v) into account, and neglecting the influence of (X,Y) on w. The division of the problems into a normal and a tangential problem is then retained. This second approximation was worked out by JOHNSON [4] for CARTER's problem, and he compared his results with the exact theory (see JOHNSON [4], fig. 7), from which it appeared that the error of the second approximation is small.

We finally observe that the problem is governed by three elastic constants, viz. G, σ, and κ. That is one less than one would expect, since in principle the four constants G^+, G^-, σ^+, σ^- can be arbitrarily chosen. We also see that G can be eliminated by introducing dimensionless tractions. So the elastic properties are taken into account by the two dimensionless parameters κ and σ, one of which we set equal to zero.

2.2. The fundamental lemma.

As we saw in the previous section, the normal and tangential problems can be formulated as the integral equations (2.11c) and (2.15a,b). They are

$$\left.\begin{aligned} u(x,y) &= \\ &= \frac{1}{\pi G} \iint_E \left[X(x',y')\left\{\frac{1-\sigma}{R} + \frac{\sigma(x-x')^2}{R^3}\right\} + Y(x',y')\,\frac{\sigma(x-x')(y-y')}{R^3}\right]dx'dy', \\ v(x,y) &= \\ &= \frac{1}{\pi G} \iint_E \left[X(x',y')\,\frac{\sigma(x-x')(y-y')}{R^3} + Y(x',y')\left\{\frac{1-\sigma}{R} + \frac{\sigma(y-y')^2}{R^3}\right\}\right]dx'dy', \end{aligned}\right\} \tag{2.18}$$

$$w(x,y) = \frac{1-\sigma}{\pi G} \iint_E Z(x',y')\,\frac{dx'dy'}{R}, \tag{2.19}$$

with

$$R = \sqrt{(x-x')^2+(y-y')^2},\quad E = \{x,y:\ x^2/a^2 + y^2/b^2 \le 1\}. \tag{2.10}$$

We will now prove the following

Fundamental Lemma:

Let

$$\left.\begin{aligned} H(x,y) &= x^k\, y^{2\ell-k}/R^{2\ell+1},\ k \text{ and } \ell \text{ positive integers},\ 2\ell \ge k; \\ J(x,y) &= \{1-(x/a)^2 - (y/b)^2\}^{-\frac{1}{2}};\ R^2 = x^2+y^2; \\ K(x,y) &= \sum_{p=0}^{M} \sum_{q=0}^{M-p} d_{pq}\, x^p y^q,\ d_{pq} \text{ arbitrary constants}; \end{aligned}\right\} \tag{2.21a}$$

$$
\left.
\begin{aligned}
&I(x,y) = \iint_E J(x',y')K(x',y')H(x-x',y-y')dx'dy',\\
&\text{then, if } (x,y) \text{ lies in } E = \{x,y:\ x^2/a^2+y^2/b^2 \le 1\},\\
&I(x,y) = \sum_{m=0}^{M}\sum_{n=0}^{M-m} a_{mn}x^m y^n,
\end{aligned}
\right\}
\quad (2.21b)
$$

that is, $I(x,y)$ is a polynomial in x,y of the same degree as $K(x,y)$.

The lemma was established by GALIN [1], ch. 2, sec. 8, in the special case that $k=\ell=0$, by means of LAME's functions. Its significance for the solution of the integral equations (2.18) and (2.19) is the following. We see that all functions of $(x-x')$ and $(y-y')$ that occur in the integrands of (2.18) and (2.19) are of the form $H(x-x',y-y')$. If we suppose that the tractions X,Y,Z are of the form $J(x,y)K(x,y)$, then it follows that the displacement differences u,v,w inside the elliptical area are polynomials in x and y of the same degree as that of $K(x,y)$. But that means that there are as many parameters in the displacement differences as there are in the tractions. There is a strong presumption x), borne out by our numerical work, that the displacement fields are independent of each other. It follows that we may invert the argument, and say that when u, v and w are given as polynomials inside E, the tractions X,Y,Z must be of the form $J(x,y)K(x,y)$. Clearly, the connection between the constants d_{pq} and a_{mn} is linear, owing to the linearity of the equations. Summarizing, we see that the lemma presumably implies that

$$
\left.
\begin{aligned}
&(u,v,w) = \sum_{m=0}^{M}\sum_{n=0}^{M-m} (a_{mn},\ b_{mn},\ c_{mn})\ x^m y^n \text{ inside } E\\
&\Longleftrightarrow (X,Y,Z)=J(x,y)G \sum_{p=0}^{M}\sum_{q=0}^{M-p} (d_{pq},e_{pq},f_{pq})x^p y^q,
\end{aligned}
\right\}
\quad (2.22)
$$

where the constants (a_{mn},b_{mn},c_{mn}) are connected with (d_{pq},e_{pq},f_{pq})

---

x) KIRCHHOFF's uniqueness theorem does not hold when the stresses go to infinity, as they do here.

by linear equations.

We now turn to the

Proof of the Lemma.

Consider a typical term of the polynomial $K(x,y)$, viz. $x^p y^q$. Then the lemma is proved, if we can show that

$$\iint_E J(x',y')x'^p y'^q\, H(x-x',y-y')dx'dy' = P_{p+q}(x,y), \tag{2.23}$$

where $P_m(x,y)$ denotes an arbitrary polynomial in x,y of degree m. We introduce polar coordinates R, ψ about the point (x,y):

$$x'-x = R\cos\psi,\; y'-y = R\sin\psi,\; dx'dy' = RdRd\psi, \tag{2.24}$$

and we introduce a new notation: $F_m(\psi)$ is an unspecified function of ψ, independent of R, x, and y, for which

$$F_m(\psi+\pi) = (-1)^m F_m(\psi). \tag{2.25}$$

For example, $\sin\psi = F_1(\psi)$, $\cos\psi = F_1(\psi)$. Multiplication of functions $F_m(\psi)$ is governed by the law that $F_m(\psi)F_n(\psi) = F_{m+n}(\psi)$. Now,

$$H(x-x',y-y') = (x-x')^k(y-y')^{2\ell-k}/R^{2\ell+1}, \text{ so,}$$

$$H(x-x',y-y') = \frac{1}{R} F_o(\psi). \tag{2.26}$$

We must write the factor $1-(x'/a)^2-(y'/b)^2$ in polar coordinates:

$$1-(x'/a)^2-(y'/b)^2 = 1 - \frac{(R\cos\psi+x)^2}{a^2} - \frac{(R\sin\psi+y)^2}{b^2} =$$

$$= \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-2R\left(\frac{x\cos\psi}{a^2}+\frac{y\sin\psi}{b^2}\right)-R^2\left(\frac{\cos^2\psi}{a^2}+\frac{\sin^2\psi}{b^2}\right)=$$

$$= - A\,\{R^2+2DR-C\} = - A\,\{(R+D)^2-C-D^2\} = A\,\{B^2-(R+D)^2\},$$

with

$$\left.\begin{aligned}
&A = \frac{\cos^2\psi}{a^2}+\frac{\sin^2\psi}{b^2} = F_o(\psi) > 0,\\
&C = \frac{1}{A}\,\{1-x^2/a^2-y^2/b^2\},\\
&D = \frac{1}{A}\left(\frac{x\cos\psi}{a^2}+\frac{y\sin\psi}{b^2}\right),\\
&B = B(\psi) = \sqrt{B^2} = \sqrt{\frac{1}{A}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+\frac{1}{A^2}\left(\frac{x\cos\psi}{a^2}+\frac{y\sin\psi}{b^2}\right)^2} =\\
&\quad = B(\pi+\psi),\\
&1 - (x'/a)^2-(y'/b)^2 = A\{B^2-(R+D)^2\}.
\end{aligned}\right\} \tag{2.27}$$

As to the limits of integration, ψ goes from 0 to 2π, since (x,y) lies inside the area of integration, and R goes from 0 to the positive zero of $1-(x'/a)^2-(y'/b)^2$, that is, to -D+B. So we get from (2.24), (2.26), and (2.27) that (2.23) becomes

$$\left.\begin{aligned} &\iint_E J(x',y')x'^p y'^q H(x-x',y-y')dx'dy' = \\ &= \int_0^{2\pi} F_o(\psi)d\psi \int_0^{B-D} \frac{(x+R\cos\psi)^p(y+R\sin\psi)^q}{\sqrt{B^2-(R+D)^2}} dR = P_{p+q}(x,y) \end{aligned}\right\} \quad (2.28)$$

where the factor $1/\sqrt{A}$ and $RH(x-x',y-y')$ have been taken together into the single term $F_o(\psi)$.

We can expand the term $(x+R\cos\psi)^p(y+R\sin\psi)^q$ to a finite double sum by means of the binomial theorem, twice applied. A typical term is $A_{ij}R^{i+j}x^{p-i}y^{q-j}\sin^i\psi\cos^j\psi$, which can be written as $R^{i+j}F_{i+j}(\psi) \times$ $\times\, x^{p-i}y^{q-j}$. Inserting this into the integral (2.28), we see that it is sufficient to prove that

$$x^{p-i}\, y^{q-j} \int_0^{2\pi} F_{i+j}(\psi)d\psi \int_0^{B-D} \frac{R^{i+j}\, dR}{\sqrt{R^2-(R+D)^2}} = P_{p+q}(x,y). \qquad (2.29)$$

Setting i+j=m, we see that (2.29) is satisfied when

$$\int_0^{2\pi} F_m(\psi)d\psi \int_0^{B-D} \frac{R^m\, dR}{\sqrt{B^2-(R+D)^2}} = P_m(x,y).$$

Now we introduce the variable t=R+D instead of R. Then, dR=dt, and the limits are from D to B:

$$\int_0^{2\pi} F_m(\psi)d\psi \int_D^B \frac{(t-D)^m dt}{\sqrt{B^2-t^2}} = P_m(x,y). \qquad (2.30)$$

We evaluate the term $(t-D)^m$ again with the binomial theorem. A typical term is $A_q t^q D^{m-q}$. If into this we introduce the value of D from (2.27), we obtain

$$A_q t^q D^{m-q} = F_o(\psi)t^q \left(\frac{x\cos\psi}{a^2} + \frac{y\sin\psi}{b^2} \right)^{m-q} .$$

Here again we evaluate the right-hand side with the binomial theorem; a typical term is

$$F_o(\psi)A_p t^q x^p y^{m-q-p}\cos^p\psi\sin^{m-p-q}\psi = F_{-m+q}(\psi)t^q x^p y^{m-p-q}.$$

Inserting this in (2.30), we get for a typical term:

$$x^p y^{m-p-q} \int_0^{2\pi} F_q(\psi)d\psi \int_D^B \frac{t^q dt}{\sqrt{B^2-t^2}} = P_m(x,y),$$

and this is satisfied if

$$\int_0^{2\pi} F_q(\psi)d\psi \int_D^B \frac{t^q dt}{\sqrt{B^2-t^2}} = P_q(x,y). \tag{2.31}$$

Now there are two possibilities: either q is odd, or q is even.

<u>q=2m+1 is odd.</u> (2.31) becomes then

$$\int_0^{2\pi} F_1(\psi)d\psi \int_D^B \frac{t^{2m+1}dt}{\sqrt{B^2-t^2}} =$$

$$= \int_0^{2\pi} F_1(\psi)d\psi \int_{D^2}^{B^2} \frac{t^m dt}{\sqrt{B^2-t}} =$$

$$= \int_0^{\pi} F_1(\psi)d\psi \left\{ \int_{D^2}^{B^2} \frac{t^m dt}{\sqrt{B^2-t}} - \int_{D^2}^{B^2} \frac{t^m dt}{\sqrt{B^2-t}} \right\} = 0,$$

since by (2.27),

$D^2(\psi+\pi) = D^2(\psi)$, $B^2(\psi+\pi) = B^2(\psi)$. So, the odd values of q do not contribute at all to the integral.

<u>q=2m is even.</u> (2.31) becomes then

$$\int_0^{2\pi} F_o(\psi)d\psi \int_D^B \frac{t^{2m}dt}{\sqrt{B^2-t^2}} = \int_0^{2\pi} F_o(\psi)d\psi \int_{D/B}^1 \frac{t^{2m}B^{2m}dt}{\sqrt{1-t^2}} =$$

$$= \int_0^{2\pi} F_o(\psi)B^{2m}d\psi \int_0^1 \frac{t^{2m}dt}{\sqrt{1-t^2}} - \int_0^{\pi} F_o(\psi)B^{2m}d\psi \left\{ \int_0^{D/B} + \int_0^{-D/B} \right\} \frac{t^{2m}dt}{\sqrt{1-t^2}},$$

and the latter two terms vanish, because $t^{2m}/\sqrt{1-t^2}$ is an even function of t. As to the first term,

$$\int_0^1 \frac{t^{2m}}{\sqrt{1-t^2}} dt$$

is a constant, so that we must consider

$$\int_0^{2\pi} F_o(\psi)B^{2m}d\psi;$$

but B^2 is a second degree polynomial in x and y, with coefficients depending upon ψ. So B^{2m} is a (2m)-degree polynomial in x and y, and (2.31) becomes

$$\int_0^{2\pi} F_{2m}(\psi)B^{2m}d\psi = P_{2m}(x,y),$$

which establishes the lemma.

2.3. DOVNOROVICH's method.

In the previous section we showed that if

$$(X,Y,Z) = GJ(x,y) \sum_{p=0}^{M} \sum_{q=0}^{M-p} (d_{pq},e_{pq},f_{pq})x^p y^q,$$

with $J(x,y) = \sqrt{1-(x/a)^2-(y/b)^2}^{\,-1}$, then and (presumably) only then (2.32)

$$(u,v,w) = \sum_{m=0}^{M} \sum_{n=0}^{M-n} (a_{mn},b_{mn},c_{mn})x^m y^n \text{ inside } E,$$

with $E = \{x,y: (x/a)^2+(y/b)^2 \le 1\}$,

where the coefficients (d,e,f) on the one hand, and (a,b,c) on the other hand are connected with each other by the integral representations (2.15a,b) and (2.11c). In order to find the equations connecting (a,b,c) and (d,e,f) explicitly, it is, of course, possible to follow exactly the road indicated by the proof of the fundamental lemma. However, we prefer the road followed by DOVNOROVICH [1] in his treatment of the normal problem. DOVNOROVICH uses the lemma only in the form proved by GALIN, that is for $H(x,y) = 1/R$. He calculates c_{mn} by differentiating the integral representation (2.11c) m times with respect to x and n times with respect to y, and then he sets x = y = 0:

$$m!n!c_{mn} = \left[\frac{\partial^{m+n}}{\partial x^m \partial y^n} \sum_{j=0}^{M} \sum_{k=0}^{M-j} c_{jk}x^j y^k\right]_{x=y=0} =$$

$$= \left[\frac{\partial^{m+n}}{\partial x^m \partial y^n} \frac{1-\sigma}{\pi G} \iint_E Z(x',y') \frac{dx'dy'}{R}\right]_{x=y=0} =$$

$$= \left[\frac{\partial^{m+n}}{\partial x^m \partial y^n} \frac{1-\sigma}{\pi} \sum_{p=0}^{M} \sum_{q=0}^{M-p} f_{pq} \iint_E J(x',y')x'^p y'^q \frac{dx'dy'}{R}\right]_{x=y=0}.$$

Since the values of p and q for which $p+q < m+n$ give rise to polynomials of a degree lower than m+n, these values do not give any contribution to c_{mn}, hence

$$m!n!c_{mn} = \\ = \frac{1-\sigma}{\pi}\left[\frac{\partial^{m+n}}{\partial x^m \partial y^n} \sum_{p+q\geq m+n, p\geq 0, q\geq 0}^{M} f_{pq} \iint_E J(x',y')x'^p y'^q \frac{dx'dy'}{R}\right]_{x=y=0}.$$

As we will prove later in this section, we may interchange differentiation and integration in this expression, so that

$$\left.\begin{aligned} &m!n!c_{mn} = \\ &= \frac{1-\sigma}{\pi} \sum_{\substack{p+q\geq m+n,\\ p\geq 0, q\geq 0}}^{M} f_{pq} \iint_E J(x,y)x'^p y'^q \left[\frac{\partial^{m+n} R^{-1}}{\partial x'^m \partial y'^n}\right]_{x'=y'=0} dxdy = \\ &= \frac{1-\sigma}{\pi} \sum_{\substack{p+q\geq m+n,\\ p\geq 0, q\geq 0}}^{M} (-1)^{m+n} f_{pq} \iint_E J(x,y)x^p y^q \left[\frac{\partial^{m+n} r^{-1}}{\partial x^m \partial y^n}\right] dxdy, \\ &r = \sqrt{x^2+y^2}. \end{aligned}\right\} \quad (2.33)$$

In exactly the same way, we find from (2.15a,b), and (2.32) that

$$\left.\begin{aligned} &m!n!a_{mn} = \\ &= \frac{1}{\pi} \sum_{\substack{p+q\geq m+n,\\ p\geq 0, q\geq 0}}^{M} (-1)^{m+n} \iint_E J(x,y)x^p y^q \quad d_{pq}\left[\left(\frac{\partial^{m+n} r^{-1}}{\partial x^m \partial y^n} - \sigma \frac{\partial^{m+n+2} r}{\partial x^{m+2} \partial y^n}\right) + \right. \\ &\qquad\qquad \left. - \sigma\, e_{pq} \frac{\partial^{m+n+2} r}{\partial x^{m+1} \partial y^{n+1}}\right] dxdy, \\ &m!n!b_{mn} = \\ &= \frac{1}{\pi} \sum_{\substack{p+q\geq m+n,\\ p\geq 0, q\geq 0}}^{M} (-1)^{m+n} \iint_E J(x,y)x^p y^q \left[- \sigma\, d_{pq} \frac{\partial^{m+n+2} r}{\partial x^{m+1} \partial y^{n+1}} + \right. \\ &\qquad\qquad \left. + e_{pq} \left(\frac{\partial^{m+n} r^{-1}}{\partial x^m \partial y^n} - \sigma \frac{\partial^{m+n+2} r}{\partial x^m \partial y^{n+2}}\right)\right] dxdy. \\ &r = \sqrt{x^2+y^2},\ E = \{x,y\colon x^2/a^2+y^2/b^2 \leq 1\}. \end{aligned}\right\} \quad (2.34)$$

The integrals

$$E_{mn}^{h;pq} = \frac{(-1)^{m+n}}{2\pi} \iint_E J(x,y)x^p y^q \frac{\partial^{m+n} r^{2h-1}}{\partial x^m \partial y^n} dxdy \quad (2.35)$$

are fairly easy to calculate; we will do that in the next section.

The remainder of this section is devoted to the proof of the validity of the equation

$$\left.\left[\frac{\partial^{m+n}}{\partial x'^m \partial y'^n} \iint_E f(x,y)H(x-x',y-y')dxdy\right]_{x'=y'=0} = \\ = (-1)^{m+n} \iint_E f(x,y) \frac{\partial^{m+n}H(x,y)}{\partial x^m \partial y^n} dxdy, \\ f(x,y) = J(x,y)x^p y^q, \; H(x,y) = (x^2+y^2)^{h-\frac{1}{2}},\right\} \tag{2.36}$$

when

$$2h+p+q-m-n > -1. \tag{2.37}$$

Proof. We divide the domain of integration into a small square

$$D = \{x,y: \; |x| < \delta, \; |y| < \delta\} \tag{2.38}$$

about the origin, and the rest E-D of E. When the point (x',y') is close enough to the origin, say

$$|x'| < \delta/2, \; |y'| < \delta/2, \tag{2.39}$$

it lies in the square D, and then all derivatives of $H(x-x',y-y')$ with respect to x' and y' exist and are continuous in E-D. Hence we may interchange differentiation and integration in E-D, so that

$$\left.\left[\frac{\partial^{m+n}}{\partial x'^m \partial y'^n} \iint_{E-D} f(x,y)H(x-x',y-y')dxdy\right]_{x'=y'=0} = \\ = (-1)^{m+n} \iint_{E-D} f(x,y) \frac{\partial^{m+n}H(x,y)}{\partial x^m \partial y^n} dxdy.\right\} \tag{2.40}$$

We will now show that the contribution of the square D to both the right hand side and the left hand side of (2.36) vanishes as $\delta \to 0$, that is

$$A = \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} f(x,y) \frac{\partial^{m+n}H(x,y)}{\partial x^m \partial y^n} dxdy \to 0 \text{ as } \delta \to 0, \tag{2.41}$$

$$B = \frac{\partial^{m+n}}{\partial x'^m \partial y'^n} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} f(x,y)H(x-x',y-y')dxdy \to 0 \text{ as } \delta \to 0 \tag{2.42}$$

Evidently this will establish (2.36).

As to (2.41), we observe that $\frac{\partial^{m+n} r^{2h-1}}{\partial x^m \partial y^n} = O\left(r^{2h-1-m-n}\right)$; moreover, $f(x,y)=J(x,y)x^p y^q = O(r^{p+q})$, so that the integrand of A is $O(r^{2h-1+p+q-m-n})$, and

$$A = O\left(\int_0^{2\pi} d\psi \int_0^{2\delta} r^{2h+p+q-m-n}\, dr\right) \to 0 \text{ as } \delta \to 0,$$

$$\text{when } 2h+p+q-m-n>-1. \qquad (2.43)$$

As to (2.42), let us consider the case that m=1, n=0. Evidently,

$$\frac{\partial}{\partial x'} \int_{-\delta}^{\delta} dy \int_{-\delta}^{\delta} f(x,y)H(x-x',y-y')dx =$$

$$= \lim_{k\to 0} \int_{-\delta}^{\delta} dy \int_{-\delta}^{\delta} f(x,y)\{H(x-x'-k,y-y')-H(x-x',y-y')\}\frac{dx}{k} =$$

$$= \lim_{k\to 0} \Big\{ \int_{-\delta}^{\delta} dy \Big[\int_{-\delta}^{\delta} \frac{f(x+k,y)-f(x,y)}{k} H(x-x',y-y')dx +$$

$$+ \frac{1}{k}\int_{-\delta-k}^{-\delta} f(x+k,y)H(x-x',y-y')dx +$$

$$- \frac{1}{k}\int_{\delta-k}^{\delta} f(x+k,y)H(x-x',y-y')dx\Big]\Big\} =$$

$$= \int_{-\delta}^{\delta} dy \int_{-\delta}^{\delta} \frac{\partial f(x,y)}{\partial x} H(x-x',y-y')dx - \int_{-\delta}^{\delta} dy \Big[f(x,y)H(x-x',y-y')\Big]_{x=-\delta}^{x=\delta},$$

or, summarizing,

$$\left.\begin{aligned} &\frac{\partial}{\partial x'} \int_{-\delta}^{\delta} dy \int_{-\delta}^{\delta} f(x,y)H(x-x',y-y')dx = \\ &\quad = \int_{\delta}^{\delta} dy \int_{-\delta}^{\delta} \frac{\partial f(x,y)}{\partial x} H(x-x',y-y')\, dx + \\ &\quad - \int_{-\delta}^{\delta} \Big[f(x,y)H(x-x',y-y')\Big]_{x=-\delta}^{x=\delta} dy. \end{aligned}\right\} \qquad (2.44)$$

We observe in passing that the right hand side of (2.44) is formally equal to $-\int f(x,y)\, \frac{\partial H}{\partial x}\, dxdy$, integrated partially. This integral, however, is not absolutely convergent when h=0, unless x'=y'=0.

The first integral on the right hand side of (2.44) is analogous to the original integral $\iint_D f(x,y)H(x-x',y-y')dxdy$; when we differentiate it further, we obtain forms analogous to (2.44). The second integral may be differentiated under the integral sign, since $H(\pm\delta-x',y-y')$ has continuous derivatives of any order with respect to x' and y', when x' and y' satisfy (2.39). So we find, by differentiating first m times with respect to x', and then n times with respect to y', that

$$\left.\begin{aligned}
&\frac{\partial^{m+n}}{\partial x'^{m}\partial y'^{n}} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} f(x,y)H(x-x',y-y')\,dxdy = \\
&\quad \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \frac{\partial^{m+n} f(x,y)}{\partial x^{m}\partial y^{n}}\, H\, dxdy \; + \\
&\quad - \sum_{i=0}^{m-1} \int_{\delta}^{\delta} \left[\frac{\partial^{i} f(x,y)}{\partial x^{i}} \; \frac{\partial^{m+n-i-1} H}{\partial x'^{m-i-1}\, \partial y'^{n}} \right]_{x=-\delta}^{x=\delta} dy \; + \\
&\quad - \sum_{i=0}^{n-1} \int_{-\delta}^{\delta} \left[\frac{\partial^{m+i} f(x,y)}{\partial x^{m}\partial y^{i}} \; \frac{\partial^{n-i-1} H}{\partial y'^{n-i-1}} \right]_{y=-\delta}^{y=\delta} dx,
\end{aligned}\right\} \qquad (2.45)$$

just as if we had integrated $(-1)^{m+n} \iint_D f(x,y) \dfrac{\partial^{m+n} H}{\partial x^{m}\partial y^{n}}\, dxdy$ partially with respect to x and y. It follows from the definition (2.36) of f(x,y) and H(x-x',y-y') that

$$\frac{\partial^{i} f(x,y)}{\partial x^{i}} = 0\left(\delta^{p+q-i}\right), \quad \frac{\partial^{m+i} f(x,y)}{\partial x^{m}\partial y^{i}} = 0\left(\delta^{p+q-m-i}\right),$$

$$\left[\frac{\partial^{m+n-i-1} H}{\partial x'^{m-i-1}\partial y'^{n}}\right]_{x=\pm\delta} = 0\left(\delta^{2h+i-m-n}\right), \quad \left[\frac{\partial^{n-i-1} H}{\partial y^{n-i-1}}\right]_{y=\pm\delta} = 0\left(\delta^{2h+i-n}\right),$$

so that the line integrals of (2.45) are all

$$0 \int_{-\delta}^{\delta} \delta^{2h+p+q-m-n}\, dy = 0\left(\delta^{2h+1+p+q-m-n}\right).$$

The surface integral of (2.45) behaves as

$$\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} \delta^{2h-1+p+q-m-n}\, dxdy = 0\left(\delta^{2h+1+p+q-m-n}\right).$$

Hence all terms of (2.45) are $0\left(\delta^{2h+1+p+q-m-n}\right)$, which vanishes as $\delta \to 0$, when $2h+p+q-m-n > -1$.

2.4. The load-displacement equations.

We saw in the previous section that when

$$
\left.
\begin{aligned}
(u,v,w) &= \sum_{m=0}^{M} \sum_{n=0}^{M-m} (a_{mn}, b_{mn}, c_{mn}) x^m y^n \text{ and} \\
(X,Y,Z) &= GJ(x,y) \sum_{p=0}^{M} \sum_{q=0}^{M-p} (d_{pq}, e_{pq}, f_{pq}) x^p y^q ,
\end{aligned}
\right\} \qquad (2.46)
$$

then is, according to (2.34), (2.33), and (2.35),

$$
a_{mn} = \frac{2}{m!n!} \sum_{\substack{p+q\geq m+n, \\ p\geq q, q\geq 0}}^{M} \left[d_{pq} \left(E^{0;pq}_{mn} - \sigma E^{1;pq}_{m+2,n} \right) - \sigma e_{pq} \, E^{1;pq}_{m+1,n+1} \right], \qquad (2.47a)
$$

$$
b_{mn} = \frac{2}{m!n!} \sum_{\substack{p+q\geq m+n, \\ p\geq 0, q\geq 0.}}^{M} \left[-\sigma d_{pq} E^{1;pq}_{m+1,n+1} + e_{pq} \left(E^{0;pq}_{mn} - \sigma E^{1;pq}_{m,n+2} \right) \right], \qquad (2.47b)
$$

$$
c_{mn} = \frac{2(1-\sigma)}{m!n!} \sum_{\substack{p+q\geq m+n, \\ p\geq 0, q\geq 0}}^{M} f_{pq} E^{0;pq}_{mn} , \qquad (2.47c)
$$

with, as we recall,

$$
\left.
\begin{aligned}
E^{h;pq}_{mn} &= \frac{(-1)^{m+n}}{2\pi} \iint_E J(x,y) x^p y^q \frac{\partial^{m+n} r^{2h-1}}{\partial x^m \partial y^n} \, dxdy, \\
&\qquad\qquad \text{when } 2h+p+q-m-n\geq 0, \\
&= 0 \qquad\qquad \text{else.} \\
r = \sqrt{x^2+y^2}, &\quad J(x,y) = \{1-(x/a)^2-(y/b)^2\}^{-\frac{1}{2}} .
\end{aligned}
\right\} \qquad (2.48)
$$

We call the equations (2.47) the <u>load-displacement equations</u>.

We can clarify the structure and the connection between (u,v,w) and (X,Y,Z) by using index notation. We set

$$
\left.
\begin{aligned}
u_i &= a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, \ldots, a_{0M}, \quad i=1 \text{ to } \tfrac{1}{2}(M+1)(M+2), \\
v_i &= b_{00}, b_{10}, b_{01}, \ldots, b_{0M}, \\
w_i &= c_{00}, c_{10}, c_{01}, \ldots, c_{0M}, \\
X_i &= d_{00}, d_{10}, d_{01}, \ldots, d_{0M}, \\
Y_i &= e_{00}, e_{10}, e_{01}, \ldots, e_{0M}, \\
Z_i &= f_{00}, f_{10}, f_{01}, \ldots, f_{0M}, \\
x_i &= 1, x, y, x^2, xy, y^2, \ldots, y^M .
\end{aligned}
\right\} (2.49)
$$

The square matrix $\frac{2}{m!n!} E^{0;pq}_{mn}$, adapted to this order, we call A_{ij}, the matrix $\frac{2}{m!n!} E^{1;pq}_{m+2,n}$ is B_{ij}, $\frac{2}{m!n!} E^{1;pq}_{m+1,n+1}$ is H_{ij}, and $\frac{2}{m!n!} E^{1;pq}_{m,n+2}$ is D_{ij}. Finally we use the summation convention: when two indices in an expression are the same, summation from 1 to $\frac{1}{2}(M+1)(M+2)$ is understood. Then we have:

$$X = GJ(x,y)X_i x_i, \quad Y = GJ(x,y)Y_i x_i, \quad Z = GJ(x,y)Z_i x_i, \qquad u = u_i x_i, \quad v = v_i x_i, \quad w = w_i x_i, \tag{2.50}$$

and the load-displacement equations are

$$\left.\begin{aligned} u_i &= \left(A_{ij}-\sigma B_{ij}\right)X_j-\sigma H_{ij}Y_j, \\ v_i &= -\sigma H_{ij}X_j + \left(A_{ij}-\sigma D_{ij}\right)Y_j, \\ w_i &= (1-\sigma)A_{ij}\ Z_j, \end{aligned}\right\} \tag{2.51}$$

so that

$$\left.\begin{aligned} u &= x_i\{\left(A_{ij}-\sigma B_{ij}\right)X_j-\sigma H_{ij}\ Y_j\}, \\ v &= x_i\{-\sigma H_{ij}X_j + \left(A_{ij}-\sigma D_{ij}\right)Y_j\}, \\ w &= x_i(1-\sigma)A_{ij}\ Z_j\ . \end{aligned}\right\} \tag{2.52}$$

We note that only x_i is position dependent. For illustration, we write out the quantities connected with Z for M=1:

$$(x_i) = (1,x,y); \ (Z_i) = (f_{00},f_{10},f_{01}); \ w_i = (c_{00},c_{10},c_{01}),$$

$$Z = GJ(1,x,y)\begin{bmatrix} f_{00} \\ f_{10} \\ f_{01} \end{bmatrix}, \quad w = (1,x,y)\begin{bmatrix} c_{00} \\ c_{10} \\ c_{01} \end{bmatrix},$$

$$\frac{1}{2(1-\sigma)}\, w = (1,x,y)\begin{bmatrix} E^{0;00}_{00} & E^{0;10}_{00} & E^{0;01}_{00} \\ E^{0;00}_{10} & E^{0;10}_{10} & E^{0;01}_{10} \\ E^{0;00}_{01} & E^{0;10}_{01} & E^{0;01}_{01} \end{bmatrix}\begin{bmatrix} f_{00} \\ f_{10} \\ f_{01} \end{bmatrix}.$$

We consider again the constants $E^{h;pq}_{mn}$ which we defined as integrals in (2.48). Since the integrand is an odd function of x when (p+m) is odd, and since the domain of integration $E = \{x,y: (x/a)^2+(y/b)^2 \leq 1\}$ is symmetric about the x-axis,

$E^{h;pq}_{mn} = 0$ when $(p+m)$ is odd. In the same way, we find that $E^{h;pq}_{mn} = 0$ when $(q+n)$ is odd. So,

$$\left.\begin{aligned} E^{h;pq}_{mn} &= \frac{(-1)^{m+n}}{2\pi} \iint_E J(x,y)x^p y^q \frac{\partial^{m+n} r^{2h-1}}{\partial x^m \partial y^n}\, dxdy, \\ &\qquad \text{when } (p+m) \text{ and } (q+n) \text{ are even, and } 2h+p+q-m-n \geq 0, \\ &= 0 \qquad \text{in all other cases.} \end{aligned}\right\} \quad (2.53)$$

The fact that $E^{h;pq}_{mn} = 0$ unless $(p+m)$ and $(q+n)$ are even, has an important practical consequence for numerical calculations. This consequence is, that the load-displacement equations for u and v, and also those for w, can be decomposed into 4 independent systems.

In order to show this, we bring out the parity of p,q,m and n by writing for p: $2p+\varepsilon$, or $2p+\varepsilon'$ as the case may be, for q: $2q+\omega$ or $2q+\omega'$, for m: $2m+\varepsilon$, or $2m+\varepsilon'$, and for n: $2n+\omega$ or $2n+\omega'$. Here, ε and ω take on the values 0 or 1 only, while ε' and ω' correspond to ε and ω by the equations $\varepsilon+\varepsilon'=1$, $\omega+\omega'=1$, so that when $\varepsilon=1$, then $\varepsilon'=0$, and when $\omega=1$, $\omega'=0$, and vice versa. Further we will consider the case that the degree M of the polynomials is given by $2K+\nu$, ($\nu=0,1$; $\nu+\nu'=1$):

$$M=2K+\nu,\ \varepsilon=0,1;\ \omega=0,1;\ \nu=0,1;\ \varepsilon+\varepsilon'=\omega+\omega'=\nu+\nu'=1. \qquad (2.54)$$

It follows from a consideration of the 8 cases $\nu=0,1$; $\varepsilon=0,1$; $\omega=0,1$, that the ranges of the summation can be represented in the formulae

$$\left.\begin{aligned} &2m+2n+\varepsilon+\omega \leq 2p+2q+\varepsilon+\omega \leq 2K+\nu \\ &\qquad \rightarrow m+n \leq p+q \leq K-\nu\varepsilon\omega+\nu'(\varepsilon'\omega'-1), \\ &2m+2n+\varepsilon+\omega \leq 2p+2q+\varepsilon'+\omega' \leq 2K+\nu \\ &\qquad \rightarrow m+n+1-\varepsilon'-\omega' \leq p+q \leq K-\nu\varepsilon'\omega'+\nu'(\varepsilon\omega-1), \end{aligned}\right\} \quad (2.55a)$$

while

$$2m+1+\varepsilon=2(m+\varepsilon)+\varepsilon',\ 2n+1+\omega=2(n+\omega)+\omega'. \qquad (2.55b)$$

So we find from (2.47):

$$\left.\begin{aligned} &\tfrac{1}{2}(2m+\varepsilon)!(2n+\omega)!\ a_{2m+\varepsilon,2n+\omega} = \\ &= \sum_{\substack{p+q=m+n,\\ p\geq 0,q\geq 0}}^{K-\nu\varepsilon\omega+\nu'(\varepsilon'\omega'-1)} d_{2p+\varepsilon,2q+\omega}\left(E^{0;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} - \sigma E^{1;2p+\varepsilon,2q+\omega}_{2(m+1)+\varepsilon,2q+\omega}\right) + \\ &- \sigma \sum_{\substack{p+q=m+n+1-\omega'-\varepsilon',\\ p\geq 0,q\geq 0}}^{K-\nu\varepsilon'\omega'+\nu'(\varepsilon\omega-1)} e_{2p+\varepsilon',2q+\omega'}\ E^{1;2p+\varepsilon',2q+\omega'}_{2(m+\varepsilon)+\varepsilon',2(n+\omega)+\omega'}, \end{aligned}\right\} \quad (2.56a)$$

$$\left.\begin{aligned}
&\tfrac{1}{2}(2m+\varepsilon')!(2n+\omega')!\ b_{2m+\varepsilon',2n+\omega'} = \\
&= -\sigma \sum_{\substack{p+q=m+n+1-\varepsilon-\omega,\\ p\geq 0,q\geq 0}}^{K-\nu\varepsilon\omega+\nu'(\varepsilon'\omega'-1)} d_{2p+\varepsilon,2q+\omega}\ E^{1;2p+\varepsilon,2q+\omega}_{2(m+\varepsilon')+\varepsilon,2(n+\omega')+\omega} + \\
&+ \sum_{\substack{p+q=m+n\\ p\geq 0,q\geq 0}}^{K-\nu\varepsilon'\omega'+\nu'(\varepsilon\omega-1)} e_{2p+\varepsilon',2q+\omega'} \left(E^{0;2p+\varepsilon',2q+\omega'}_{2m+\varepsilon',2n+\omega'} - \sigma E^{1;2p+\varepsilon',2q+\omega'}_{2m+\varepsilon',2(n+1)+\omega'}\right),
\end{aligned}\right\} \tag{2.56b}$$

$$\left.\begin{aligned}
&\tfrac{1}{2}(2m+\varepsilon)!(2n+\omega)!\ c_{2m+\varepsilon,2n+\omega} = \\
&= (1-\sigma) \sum_{\substack{p+q=m+n,\\ p\geq 0,q\geq 0}}^{K-\nu\varepsilon\omega+\nu'(\varepsilon'\omega'-1)} f_{2p+\varepsilon,2q+\omega}\ E^{0;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}\ .
\end{aligned}\right\} \tag{2.56c}$$

We see immediately from these equations that the systems (2.56a) and (2.56b) taken together form a closed system of equations for each of the four possible choices for (ε,ω), viz. $(\varepsilon,\omega)=(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. The same can be said of the system (2.56c). Moreover, when $\sigma=0$, there is no longer any interaction between Y and u, and between X and v, sc that the equations (2.56a) can be solved independently of (2.56b); in fact, (2.56a) and (2.56b) get the same form as (2.56c) with $\sigma=0$.

After these general considerations, we will determine $E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}$ in the next subsections.

2.41. A differentiation formula.

In the present subsection, we derive the following differentiation formula:

$$\frac{\partial^{m+n}(x^2+y^2)^{\alpha}}{\partial x^m \partial y^n} =$$

$$= \sum_{k\geq m/2}^{m} \sum_{\ell\geq n/2}^{n} \frac{(-1)^{k+\ell}\ (-\alpha)_{k+\ell}\ m!n!}{(m-k)!(n-\ell)!(2k-m)!(2\ell-n)!}(x^2+y^2)^{\alpha-k-\ell}(2x)^{2k-m}(2y)^{2\ell-n}, \tag{2.57}$$

in which we use the notation $(z)_j$:

$$(z)_j = \frac{\Gamma(z+j)}{\Gamma(z)}\ ;\ (z)_0=1;\ (z)_j=z(z+1)\dots(z+j-1),\ j=1,2,3,\dots \tag{2.58}$$

Proof. We expand $\{(x+u)^2+(y+v)^2\}^\alpha$ about (x^2+y^2). According to TAYLOR's theorem, we have that

$$H \equiv \{(x+u)^2+(y+v)^2\}^\alpha = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{u^m v^n}{m!n!} \frac{\partial^{m+n}(x^2+y^2)^\alpha}{\partial x^m \partial y^n} . \qquad (2.59)$$

This expansion has a radius of convergence which differs from zero, when $(x^2+y^2) \neq 0$.

On the other hand, we can expand H by means of the binomial theorem:

$$H \equiv \{(x+u)^2+(y+v)^2\}^\alpha = \{(x^2+y^2)+(2xu+u^2+2yv+v^2)\}^\alpha =$$

$$= (x^2+y^2)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k(-\alpha)_k}{k!} \frac{(2xu+u^2+2yv+v^2)^k}{(x^2+y^2)^k} =$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{(-1)^k(-\alpha)_k}{(k-\ell)!\ell!} (x^2+y^2)^{\alpha-k} (2xu+u^2)^{k-\ell} (2yv+v^2)^\ell .$$

In this double sum, we interchange the summation. The summation ranges are then $0\leq\ell<\infty$, $\ell\leq k<\infty$. Then, we replace k by $k+\ell$, which gives us an expression for H which is symmetric in k and ℓ:

$$H = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k+\ell}(-\alpha)_{k+\ell}}{k!\ell!} (x^2+y^2)^{\alpha-k-\ell} (2xu+u^2)^k (2yv+v^2)^\ell =$$

$$= \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{\ell} \frac{(-1)^{k+\ell}(-\alpha)_{k+\ell} u^{k+m} v^{\ell+n}}{(k-m)!(\ell-n)!n!m!} (x^2+y^2)^{\alpha-k-\ell} (2x)^{k-m} (2y)^{\ell-n} .$$

In order to get $u^m v^n$ in this sum, we replace m by $m-k$, and n by $n-\ell$:

$$H = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=k}^{2k}\sum_{n=\ell}^{2\ell} \frac{(-1)^{k+\ell}(-\alpha)_{k+\ell} u^m v^n}{(2k-m)!(m-k)!(2\ell-n)!(n-\ell)!} \times$$

$$\times (x^2+y^2)^{\alpha-k-\ell} (2x)^{2k-m} (2y)^{2\ell-n} .$$

We bring the summation over m and n in front. The range of summation of k and m was: $0\leq k<\infty$, $k\leq m\leq 2k$; this becomes $0\leq m\leq\infty$, $\frac{1}{2}m\leq k\leq m$. So,

$$H = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k\geq m/2}^{m}\sum_{\ell\geq n/2}^{n} \frac{(-1)^{k+\ell}(-\alpha)_{k+\ell}(x^2+y^2)^{\alpha-k-\ell}}{(m-k)!(n-\ell)!(2k-m)!(2\ell-n)!} \times$$

$$\times u^m v^n (2x)^{2k-m} (2y)^{2\ell-n} . \qquad (2.60)$$

Comparing this with (2.59), we see, that indeed

$$\frac{\partial^{m+n}(x^2+y^2)^\alpha}{\partial x^m \partial y^n} =$$

$$= \sum_{k\geq m/2}^{m} \sum_{\ell\geq n/2}^{n} \frac{(-1)^{k+\ell}(-\alpha)_{k+\ell}\, m!\, n!}{(m-k)!(n-\ell)!(2k-m)!(2\ell-n)!} (x^2+y^2)^{\alpha-k-\ell}(2x)^{2k-m}(2y)^{2\ell-n},$$

as we set out to prove.

2.42. The coefficients of the load-displacement equations as finite sums of complete elliptic integrals.

We use the differentiation formula (2.57) to calculate the integrals

$$E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} =$$

$$= \frac{(-1)^{\varepsilon+\omega}}{2\pi} \iint_E J(x,y)x^{2p+\varepsilon}y^{2q+\omega} \frac{\partial^{2m+2n+\varepsilon+\omega} r^{2h-1}}{\partial x^{2m+\varepsilon}\partial y^{2n+\omega}} \, dxdy, \qquad (2.61)$$

where

$$\varepsilon=0,1; \quad \omega=0,1; \quad h+p+q-m-n>-\tfrac{1}{2} \qquad (2.62)$$

see (2.53), in which the coefficients of the load-displacement equations (2.56) are expressed.

We call $|e|$ the excentricity of the contact ellipse $(x/a)^2+(y/b)^2 = 1$, $0\leq|e|\leq1$; $g = \sqrt{1-e^2}$ is the ratio of the axes. When a is the minor semi-axis, we take $e\geq0$. We will denote the minor semi-axis by s:

$$\left.\begin{array}{ll} e\geq0: \ s=a=gb\leq b=s/g, & J=\{1-(x/s)^2-(gy/s)^2\}^{-\frac{1}{2}}, \\ e\leq0: \ s=b=ga\leq a=s/g, & J=\{1-(gx/s)^2-(y/s)^2\}^{-\frac{1}{2}}, \\ g = \sqrt{1-e^2}, \quad |e| = \sqrt{1-g^2}. & \end{array}\right\} \qquad (2.63)$$

We interchange in (2.61) x and y, p and q, m and n, ε and ω. Taking (2.63) into account, we see that

$$E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(e) = E^{h;2q+\omega,2p+\varepsilon}_{2n+\omega,2m+\varepsilon}(-e). \qquad (2.64)$$

So, without loss of generality, we consider the case of $e\geq0$ only.

We substitute the differentiation formula (2.57) into (2.61). This gives:

$$E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} =$$
$$= \frac{1}{2\pi} \sum_{k=m+\varepsilon}^{2m+\varepsilon} \sum_{\ell=n+\omega}^{2n+\omega} \frac{(-1)^{k+\ell+\varepsilon+\omega}(\frac{1}{2}-h)_{k+\ell}(2m+\varepsilon)!(2n+\omega)!}{(2m+\varepsilon-k)!(2n+\omega-\ell)!(2k-2m-\varepsilon)!(2\ell-2n-\omega)!} \times$$
$$\times\, 2^{2k+2\ell-2m-2n-\varepsilon-\omega} \iint_E J(x,y)x^{2k+2p-2m}y^{2\ell+2q-2n}(x^2+y^2)^{h-\frac{1}{2}-k-\ell}\, dxdy,$$
with $J(x,y) = \{1-(x/s)^2-(gy/s)^2\}^{-\frac{1}{2}}$. (2.65)

In the double integral (2.65) we introduce polar coordinates:

$$x = sr\cos\psi, \quad y = sr\sin\psi, \quad dxdy = s^2 r dr d\psi.$$

The form $J(x,y)$ becomes

$$J(x,y)=\{1-r^2\cos^2\psi-r^2g^2\sin^2\psi\}^{-\frac{1}{2}} = \{1-D^2r^2\}^{-\frac{1}{2}}, \; D = \sqrt{1-e^2\sin^2\psi}. \qquad (2.66)$$

The integration is taken over all points x and y, for which $J(x,y)$ is real. That is, the limits are in polar coordinates: $0\leq\phi<2\pi$, $0\leq r\leq 1/D$. If we set $2k+2p-2m=2i$, $2\ell+2q-2n=2j$ in (2.65), we see that a typical integral of (2.65) becomes

$$I = s^{2d+1} \int_0^{2\pi} d\psi \int_0^{1/D} \frac{\cos^{2i}\psi\sin^{2j}\psi r^{2d}dr}{\sqrt{1-D^2r^2}},$$
$$i = k+p-m, \; j = \ell+q-n, \; d = h+p+q-m-n. \qquad (2.67)$$

Changing the variable to $t = D^2r^2$, with $dr = \dfrac{dt}{2D\sqrt{t}}$, we obtain

$$I = s^{2d+1} \int_0^{2\pi} \frac{\cos^{2i}\psi\sin^{2j}\psi d\psi}{2D^{2d+1}} \int_0^1 t^{d-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\, dt. \qquad (2.68)$$

The integral over t in (2.68) is a complete Beta function,

$$B(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt = \Gamma(x)\Gamma(y)/\Gamma(x+y). \qquad (2.69)$$

As to ψ, we may restrict ourselves to the interval $0\leq\psi\leq\pi/2$, owing to the symmetry of the integrand. So we get from (2.66), (2.68), and (2.69), that

$$I = \iint_E J(x,y)\,\frac{x^{2i}\,y^{2j}\,dxdy}{(x^2+y^2)^{i+j-d+\frac{1}{2}}} =$$
$$= \frac{2s^{2d+1}\Gamma(d+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(d+1)} \int_0^{\pi/2} \frac{\cos^{2i}\psi\sin^{2j}\psi d\psi}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}}. \qquad (2.70)$$

This is a complete elliptic integral of a general type, which can, in principle, be reduced to a combination of elliptic integrals of the first and second kind. We substitute (2.70) into (2.65), setting $i=k+p-m$, $j=\ell+q-n$. Then,

$$\left.\begin{aligned}
&E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(|e|) = E^{h;2q+\omega,2p+\varepsilon}_{2n+\omega,2m+\varepsilon}(-|e|) = \\
&= \frac{1}{2\pi}\sum_{k=m+\varepsilon}^{2m+\varepsilon}\ \sum_{\ell=n+\omega}^{2n+\omega} \frac{(-1)^{k+\ell+\varepsilon+\omega}(\frac{1}{2}-h)_{k+\ell}(2m+\varepsilon)!(2n+\omega)!4^{k+\ell-m-n}2^{-\varepsilon-\omega}}{(2m+\varepsilon-k)!(2n+\omega-\ell)!(2k-2m-\varepsilon)!(2\ell-2n-\omega)!}\times \\
&\quad\times \frac{2s^{2d+1}\Gamma(d+\frac{1}{2})\Gamma(\frac{1}{2})}{d!}\int_0^{\pi/2}\frac{(\cos^2\psi)^{k+p-m}(\sin^2\psi)^{\ell+q-n}d\psi}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}}, \\
&\quad d = h+p+q-m-n\geq 0.
\end{aligned}\right\} \tag{2.71}$$

We replace k by $k+m+\varepsilon$, ℓ by $\ell+n+\omega$. The limits of summation then become $0\leq k\leq m$, $0\leq \ell\leq n$. Making use of the formulae of the Gamma function

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \pi/\cos(\pi z),\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi},\ (z)_j = \frac{\Gamma(z+j)}{\Gamma(z)}, \tag{2.72}$$

it is easy to see that $(\frac{1}{2}-h)_{k+\ell}\Gamma(d+\frac{1}{2})\Gamma(\frac{1}{2})$ in (2.71) becomes

$$(\tfrac{1}{2}-h)_{k+\ell+m+n+\varepsilon+\omega}\Gamma(d+\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2}-h+m+n+k+\ell+\varepsilon+\omega)\Gamma(\frac{1}{2})\pi}{\Gamma(\frac{1}{2}-h)\Gamma(\frac{1}{2}-d)\cos\pi d} =$$

$$= \frac{\Gamma(\frac{1}{2}-d+p+q+k+\ell+\varepsilon+\omega)}{\Gamma(\frac{1}{2}-d)}\ \frac{\cos\pi h}{\cos\pi d}\ \frac{\Gamma(\frac{1}{2}+h)}{\Gamma(\frac{1}{2})}\ \pi =$$

$$= (-1)^{p+q-m-n}\pi(\tfrac{1}{2}-d)_{p+q+k+\ell+\varepsilon+\omega}(\tfrac{1}{2})_h\ .$$

So, (2.71) becomes

$$E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(e) =$$

$$=(\tfrac{1}{2})_h(-2)^{\varepsilon+\omega}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{4^{k+\ell}(2m+\varepsilon)!(2n+\omega)!s^{2d+1}}{(m-k)!(n-\ell)!(2k+\varepsilon)!(2\ell+\omega)!}\,I(d,k+p+\varepsilon,\ell+q+\omega,e), \tag{2.73}$$

with

$$\left.\begin{aligned}
&I(d,i,j,|e|)=I(d,j,i,-|e|)=\frac{1}{d!}(\tfrac{1}{2}-d)_{i+j}\int_0^{\pi/2}\frac{(-\cos^2\psi)^i(-\sin^2\psi)^j d\psi}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}}, \\
&\quad \frac{1}{d!} = 0 \text{ when } d = -1, -2, -3, \ldots
\end{aligned}\right\} \tag{2.74}$$

which is valid when $d = h+p+q-m-n>-\frac{1}{2}$. When h is an integer, as it is in the load displacement equations, d is also an integer, and then (2.73) is a finite sum of complete elliptic integrals of a general type which can be reduced to complete elliptic integrals of the first and second kind.

It is useful for the purpose of numerical calculations, to know beforehand what elliptic integrals (2.74) actually occur in the load-displacement equations (2.56). Let the degree of the polynomials (2.46) be 2k+ν, with ν = 0 or 1. Then it can be shown that

$$\left.\begin{array}{l} M=2k+\nu,\ \nu=0,1 \rightarrow 0 \le d \le k,\ d \le i+j \le 2k+\nu-d,\ i \ge 0,\ j \ge 0, \\ \quad \text{for } w \text{ (eq. 2.56c) and for } u,v \text{ when } \sigma = 0 \text{ (eq. 2.56a,b)}, \end{array}\right\} \quad (2.75)$$

and

$$\left.\begin{array}{l} M=2k+\nu,\ \nu=0,1 \rightarrow 0 \le d \le k,\ d \le i+j \le 2k+1+\nu-d,\ i \ge 0,\ j \ge 0, \\ \quad \text{for } u,v \text{ (eq. 2.56a,b)},\ \sigma \ne 0. \end{array}\right\} \quad (2.76)$$

2.43. Transformation to another metric.

We will consider the case that we transform the coordinate system (x,y,z) to another coordinate system $(\bar{x},\bar{y},\bar{z})$ with the same origin and axes, but with another metric:

$$\bar{x} = \lambda x,\ \bar{y} = \lambda y,\ \bar{z} = \lambda z,\ \bar{s} = \lambda s,\ (\lambda \text{ constant}). \qquad (2.77)$$

We distinguish quantities taken with respect to $(\bar{x},\bar{y},\bar{z})$ from the corresponding quantities in (x,y,z) by a bar over the letter. Clearly, we have

$$\left.\begin{array}{l} (\bar{u}^{\pm},\bar{v}^{\pm},\bar{w}^{\pm}) = \lambda(u^{\pm},v^{\pm},w^{\pm}), \\ (\bar{u},\bar{v},\bar{w}) \quad = \lambda(u,v,w), \\ (\bar{X},\bar{Y},\bar{Z})/\bar{G} = (X,Y,Z)/G,\ \bar{G} = G/\lambda^2. \end{array}\right\} \qquad (2.78)$$

Also,

$$\left.\begin{array}{l} J(\bar{x},\bar{y}) = \sqrt{1-(\bar{x}/\bar{a})^2-(\bar{y}/\bar{b})^2}^{\,-1} = \sqrt{1-(x/a)^2-(y/b)^2}^{\,-1} = J(x,y), \\ \quad \bar{g} = g,\ \bar{e} = e. \end{array}\right\} \quad (2.79)$$

It is easy to see that

$$\begin{aligned}(\bar{u},\bar{v},\bar{w}) &= \sum_{m=0}^{M}\sum_{n=0}^{M-m} (\bar{a}_{mn},\bar{b}_{mn},\bar{c}_{mn})\bar{x}^m\,\bar{y}^n = \\ &= \sum_{m=0}^{M}\sum_{n=0}^{M-m} \lambda^{m+n}\,(\bar{a}_{mn},\bar{b}_{mn},\bar{c}_{mn})x^m\,y^n = \\ &= \lambda(u,v,w) = \sum_{m=0}^{M}\sum_{n=0}^{M-m} \lambda(a_{mn},b_{mn},c_{mn})x^m\,y^n,\end{aligned}$$

from which it follows that

$$\bar{a}_{mn} = \lambda^{1-m-n}\,a_{mn},\ \bar{b}_{mn} = \lambda^{1-m-n}\,b_{mn},\ \bar{c}_{mn} = \lambda^{1-m-n}\,c_{mn}, \qquad (2.80)$$

and it follows in the same way from

$$(\bar{X},\bar{Y},\bar{Z})/\bar{G} = J(\bar{x},\bar{y}) \sum_{p=0}^{M} \sum_{q=0}^{M-p} (\bar{d}_{pq},\bar{e}_{pq},\bar{f}_{pq})\bar{x}^p \bar{y}^q =$$

$$= (X,Y,Z)/G = J(x,y) \sum_{p=0}^{M} \sum_{q=0}^{M-p} (d_{pq},e_{pq},f_{pq})\, x^p y^q,$$

that

$$\bar{d}_{pq} = \lambda^{-p-q}\, d_{pq},\ e_{pq} = \lambda^{-p-q}\, e_{pq},\ \bar{f}_{pq} = \lambda^{-p-q}\, f_{pq}. \tag{2.81}$$

From (2.73) and (2.74) we see with the aid of (2.77) and (2.79) that

$$\bar{E}^{h;pq}_{mn} = \lambda^{2d+1}\, E^{h;pq}_{mn},\ 2d = 2h+p+q-m-n. \tag{2.82}$$

If (a_{mn},b_{mn},c_{mn}) and (d_{pq},e_{pq},f_{pq}) are such that the (unbarred) load-displacement equations are satisfied, we see immediately from (2.47) that the barred load-displacement equations are satisfied by

$$(\bar{a}_{mn},\bar{b}_{mn},\bar{c}_{mn}) = \lambda^{1-m-n}\,(a_{mn},b_{mn},c_{mn}),$$

$$(\bar{d}_{pq},\bar{e}_{pq},\bar{f}_{pq}) = \lambda^{-p-q}\,(d_{pq},e_{pq},f_{pq}),$$

that is, by the same parameters as in (2.80) and (2.81). So, solving the load-displacement equations for one value of λ, means solving them for all λ.

3. Special cases of the load-displacement equations.

In section 3.1 of the present chapter, we develop the theory of the load-displacement equations further. In fact, we will study the special case that the traction behaves as $\sqrt{1-(x/a)^2-(y/b)^2}$ as one approaches the edge of the contact area, rather than as $\sqrt{1-(x/a)^2-(y/b)^2}^{\,-1}$, as we had in chapter 2, see eq. (2.32). This is of importance in some applications of which we will name the normal problem of HERTZ, which is treated in 3.221, and the tangential problem of CATTANEO [1], and MINDLIN [1], which is treated in section 3.222. Since for a general polynomial displacement the traction goes to infinity at the edge, the demand that the traction must vanish constitutes a restraint on the displacement, in other terms, the displacement must have a special form in order to meet it. In the HERTZ case this special form results from the adaptation of form and size of the contact ellipse; similarly, in the MINDLIN-CATTANEO problem of section 3.222, and in CARTER's [1] problem, the area of adhesion is so adapted.

One can perhaps say that in tangential problems in which slip is actually present, but is neglected in the calculation, the load-displacement equations of section 2.4 must be used: the infinity of the traction at the edge of the contact area indicates an area of slip. This is the case, at any rate, in the MINDLIN-CATTANEO problem without slip (sec. 3.212), in DE PATER's [1] treatment of the problem of the rolling contact between two cylinders with parallel axes with infinitesimal longitudinal creepage, and in the treatment of the problem of rolling contact with infinitesimal creepage and spin of section 4.3. In that section, the interpretation of the traction singularity is treated more fully. In normal problems, the pressure singularity can indicate a sharp edge, as is the case in the problem of an elliptical die pressed into a half-space, see section 3.211.

If in the tangential problems slip is not neglected, as we have in sec. 3.222, the MINDLIN-CATTANEO problem with slip, without twist, and in the theory of rolling with arbitrary creepage and spin, chapter 5, the tangential traction generally vanishes at the edge of the contact area. For the normal pressure distribution will mostly be Hertzian, and the friction law demands that $|(X,Y)| \leq \mu Z$. So X and

Y must also vanish at the edge of the contact area, and at least as fast as the normal load Z.

3.1. The load-displacement equations, when the surface tractions vanish at the edge of the contact area.

As we pointed out in section 3, the demand of vanishing traction at the edge of the contact area E constitutes a restraint on the surface displacement differences (u,v,w).

We had found in sec. 2.2 (see 2.32) that when

$$(X,Y,Z) = G\{1-x^2/a^2-y^2/b^2\}^{-\frac{1}{2}} \sum_{p=0}^{M+2} \sum_{q=0}^{M+2-p} \left(d'_{pq}, e'_{pq}, f'_{pq}\right) x^p y^q, \tag{3.1}$$

then

$$(u,v,w) = \sum_{m=0}^{M+2} \sum_{n=0}^{M+2-m} (a_{mn}, b_{mn}, c_{mn}) x^m y^n. \tag{3.2}$$

Now, the tractions must vanish at the edge of the contact area. This means that the constants $\left(d'_{pq}, e'_{pq}, f'_{pq}\right)$ must be so, that

$$\sum_{p=0}^{M+2} \sum_{q=0}^{M+2-p} \left(d'_{pq}, e'_{pq}, f'_{pq}\right) x^p y^q$$

is divisible by $\{1-(x/a)^2-(y/b)^2\}$. That means that

$$\left.\begin{aligned}
(X,Y,Z) &= GJ(x,y) \sum_{p=0}^{M+2} \sum_{q=0}^{M+2-p} \left(d'_{pq}, e'_{pq}, f'_{pq}\right) x^p y^q \\
&= GJ(x,y) \sum_{p=0}^{M} \sum_{q=0}^{M-p} \left(d_{pq}, e_{pq}, f_{pq}\right) \{1-(x/a)^2-(y/b)^2\} x^p y^q \\
&= G\sqrt{1-(x/a)^2-(y/b)^2} \sum_{p=0}^{M} \sum_{q=0}^{M-p} \left(d_{pq}, e_{pq}, f_{pq}\right) x^p y^q \\
&= GJ(x,y) \sum_{p=0}^{M} \sum_{q=0}^{M-p} \left(d_{pq}, e_{pq}, f_{pq}\right) \left(x^p y^q - \frac{1}{a^2} x^{p+2} y^q - \frac{1}{b^2} x^p y^{q+2}\right), \\
J(x,y) &= \{1-(x/a)^2-(y/b)^2\}^{-\frac{1}{2}} .
\end{aligned}\right\} \tag{3.3}$$

Comparing (3.2) and (3.3), we see that there are more constants (a_{mn}, b_{mn}, c_{mn}) in (3.2) than there are constants (d_{pq}, e_{pq}, f_{pq}) in (3.3). So, the matrix of the load-displacement equations is no longer square.

We seek the connection between (a_{mn}, b_{mn}, c_{mn}) on the one hand, and (d_{pq}, e_{pq}, f_{pq}) on the other hand. For that purpose we define

$$F^{h;pq}_{mn} = E^{h;pq}_{mn} - \frac{1}{a^2} E^{h;p+2,q}_{mn} - \frac{1}{b^2} E^{h;p,q+2}_{mn}. \qquad (3.4)$$

We note that by (2.53) $E^{h;pq}_{mn} = 0$ when $2h+p+q-m-n<0$, $(h=0,1)$, but that for $2h+p+q-m-n = -2$, $E^{h;p+2,q}_{mn}$ and $E^{h;p,q+2}_{mn}$ do not vanish for all values of p,q,m,n. Keeping this in mind, we see from (3.3), (3.2), (3.4), and (2.47) that

$$\left.\begin{aligned}
a_{mn} &= \frac{2}{m!n!} \sum^{M}_{\substack{p+q\geq m+n-2,\\ p\geq 0,q\geq 0}} \left\{d_{pq}\left(F^{0;pq}_{mn} - \sigma\, F^{1;pq}_{m+2,n}\right) - \sigma\, e_{pq}\, F^{1;pq}_{m+1,n+1}\right\},\\
b_{mn} &= \frac{2}{m!n!} \sum^{M}_{\substack{p+q\geq m+n-2,\\ p\geq 0,q\geq 0}} \left\{e_{pq}\left(F^{0;pq}_{mn} - \sigma\, F^{1;pq}_{m,n+2}\right) - \sigma d_{pq}\, F^{1;pq}_{m+1,n+1}\right\},\\
c_{mn} &= \frac{2(1-\sigma)}{m!n!} \sum^{M}_{\substack{p+q\geq m+n-2,\\ p\geq 0,q\geq 0}} f_{pq}\, F^{0;pq}_{mn}\,.
\end{aligned}\right\} \qquad (3.5)$$

We will now calculate $F^{h;pq}_{mn}$. We see from (3.4) and (2.53) that

$$F^{h;pq}_{mn} = 0 \text{ when } 2h+p+q-m-n = -3, -4, -5, \ldots, \qquad (3.6a)$$

and we note that (p+2) and (q+2) have the same parity as p and q, respectively, so that it follows from (3.4) and (2.53) that

$$F^{h;pq}_{mn} = 0 \text{ unless both } (p+m) \text{ and } (q+n) \text{ are even.} \qquad (3.6b)$$

Hence, the load-displacement equations can again be decomposed into 4 sets. Further, by (2.64) we have from (3.4) that

$$F^{h;pq}_{mn}(|e|) = E^{h;pq}_{mn}(|e|) - (1/s^2)E^{h;p+2,q}_{mn}(|e|) - (g^2/s^2)E^{h;p,q+2}_{mn}(|e|), \qquad (3.7a)$$

$$F^{h;qp}_{nm}(-|e|) = E^{h;qp}_{nm}(-|e|) - (1/s^2)E^{h;q,p+2}_{nm}(-|e|) - (g^2/s^2)E^{h;q+2,p}_{nm}(-|e|), \qquad (3.7b)$$

where s denotes the minor semi-axis, and g is the ratio of the axes min (a/b,b/a). Since $E^{h;pq}_{mn}(e) = E^{h;qp}_{nm}(-e)$ according to (2.64), it follows from (3.7) that

$$F^{h;pq}_{mn}(e) = F^{h;qp}_{nm}(-e). \tag{3.8}$$

So, by (3.6) and (3.8), we only have to calculate $F^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(|e|)$. We consider a change of metric as described in sec. 2.43. It is easy to see from (3.2), (3.3), (3.5), and (3.4), and from (2.32) and (2.47) that the analysis of 2.43 remains valid in the present case of zero stress at the edge of the contact area, so that the effect of a change of metric here is also described by (2.80), (2.81), and (2.82), if we read F for E. So, we have to set up the load-displacement equations for one metric only.

We see from (3.4) that the $E^{h;pq}_{mn}$ occurring in the expression for $F^{h;pq}_{mn}$ all have the same h, m, and n. So in substituting the $E^{h;pq}_{mn}$ from (2.73), we can bring the double summation outside the brackets. Then we have for $e \geq 0$:

$$\left.\begin{aligned}
&F^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} = \\
&= E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} - (1/s^2)E^{h;2(p+1)+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega} - (g^2/s^2)E^{h;2p+\varepsilon,2(q+1)+\omega}_{2m+\varepsilon,2n+\omega} = \\
&= (\tfrac{1}{2})_h(-2)^{\varepsilon+\omega}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{4^{k+\ell}(2m+\varepsilon)!(2n+\omega)!\; s^{2d+1}}{(m-k)!(n-\ell)!(2k+\varepsilon)!(2\ell+\omega)!}\{I(d,k+p+\varepsilon,\ell+q+\omega,e)+ \\
&-I(d+1,k+p+1+\varepsilon,\ell+q+\omega,e)-g^2I(d+1,k+p+\varepsilon,\ell+q+1+\omega,e)\}, \\
&e\geq 0,\quad d=h+p+q-m-n.
\end{aligned}\right\} \tag{3.9}$$

We define

$$\left.\begin{aligned}
J(d,i,j,|e|) &= I(d,i,j,|e|) - I(d+1,i+1,j,|e|) - g^2I(d+1,i,j+1,|e|), \\
J(d,j,i,-|e|) &= I(d,j,i,-|e|) - I(d+1,j,i+1,-|e|) - g^2I(d+1,j+1,i,-|e|),
\end{aligned}\right\} \tag{3.10}$$

so that

$$J(d,i,j,e)=J(d,j,i,-e), \tag{3.11}$$

and from (3.8),(3.9),(3.10) and (3.11) it follows that

$$\left.\begin{aligned}
&F^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(e) = F^{h;2q+\omega,2p+\varepsilon}_{2n+\omega,2m+\varepsilon}(-e) = \\
&= (\tfrac{1}{2})_h(-2)^{\varepsilon+\omega}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{4^{k+\ell}(2m+\varepsilon)!(2n+\omega)!\; s^{2d+1}}{(m-k)!(n-\ell)!(2k+\varepsilon)!(2\ell+\omega)!}J(d,k+p+\varepsilon,\ell+q+\omega,e). \\
&F^{h;pq}_{mn} = 0 \text{ unless } (p+m),\ (q+n) \text{ are both even and } d=h+p+q-m-n\geq -1.
\end{aligned}\right\} \tag{3.12}$$

Comparing this with (2.73), we see that $F^{h;pq}_{mn}$ and $E^{h;pq}_{mn}$ have exactly the same form, the only difference is that F has J-functions where E has I-functions.

We calculate $J(d,i,j,|e|)$ from (3.10) and (2.74).

$$J\left(d,i,j,|e|\right)= I\left(d,i,j,|e|\right)- I\left(d+1,i+1,j,|e|\right)-g^2 I\left(d+1,i,j+1,|e|\right) =$$

$$= \frac{1}{d!}\left(\tfrac{1}{2}-d\right)_{i+j}\int_0^{\pi/2}\frac{(-\cos^2\psi)^i(-\sin^2\psi)^j\,d\psi}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}} - \frac{1}{(d+1)!}\left(\tfrac{1}{2}-d-1\right)_{i+j+1}\times$$

$$\times\int_0^{\pi/2}\frac{(-\cos^2\psi)^i(-\sin^2\psi)^j\,d\psi}{(1-e^2\sin^2\psi)^{d+3/2}}\left(-\cos^2\psi-g^2\sin^2\psi\right).$$

Since $1/d!$ must be interpreted as zero when $d = -1$, we can write $1/d! = \frac{d+1}{(d+1)!}$; further, $\cos^2\psi+g^2\sin^2\psi = 1-e^2\sin^2\psi$, and finally $(\frac{1}{2}-d-1)_{i+j+1} = (\frac{1}{2}-d-1)(\frac{1}{2}-d)_{i+j}$, so that

$$\left.\begin{aligned} J\left(d,i,j,|e|\right) &= \frac{2d+2-2d-1}{2(d+1)!}\left(\tfrac{1}{2}-d\right)_{i+j}\int_0^{\pi/2}\frac{(-\cos^2\psi)^i(-\sin^2\psi)^j}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}}\,d\psi \\ &= \tfrac{1}{2}\,\frac{1}{(d+1)!}\left(\tfrac{1}{2}-d\right)_{i+j}\int_0^{\pi/2}\frac{(-\cos^2\psi)^i(-\sin^2\psi)^j}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}}\,d\psi, \\ J\left(d,i,j,|e|\right) &= J\left(d,j,i,-|e|\right),\ 1/(d+1)! = 0 \text{ when } d = -2,-3\ \ldots \end{aligned}\right\} \tag{3.13}$$

Comparing this with (2.74), we see that

$$I(d,i,j,e) = 2(d+1)J(d,i,j,e), \tag{3.14}$$

so that we find from (3.12) and (2.73), that

$$E^{h;pq}_{mn} = 2(d+1)F^{h;pq}_{mn}\ ,\ 2d = 2h+p+q-m-n, \tag{3.15}$$

which means that the coefficients of the load-displacement equations for an infinite traction at the edge of the contact area can be found by multiplying the corresponding coefficient of the load-displacement equation with zero traction at the edge with $2(d+1)$.

It is useful for the purpose of numerical calculations to know beforehand which elliptical integrals (3.13) occur. When the degree of the traction polynomial is $M = 2K+\nu$, $\nu = 0$ or 1, it can be shown that

$M=2K+\nu$: w, and (u,v) when $\sigma=0$: $-1\le d\le K$, $\max(0,d)\le i+j\le 2K+\nu-d$ (3.16a)

$M=2K+\nu$: (u,v) when $\sigma\neq 0$: $-1\le d\le K$, $\max(0,d)\le i+j\le 2K+1+\nu-d$. (3.16b)

3.2. Examples of the use of the load-displacement equations. A list of the functions $J(d,i,j,e)$ and $F^{h;pq}_{mn}$.

In the present section we give a few examples of the use of the load-displacement equations. First we will give a list of the elliptic integrals out of which the $F^{h;pq}_{mn}$ are formed, and a list of these coefficients themselves. We define with JAHNKE & EMDE [2]:

$$\underline{K} = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-e^2\sin^2\psi}} \; , \quad \underline{E} = \int_0^{\pi/2} \sqrt{1-e^2\sin^2\psi}\; d\psi \; , \tag{3.17a}$$

$$\underline{C} = \int_0^{\pi/2} \frac{\sin^2\psi\cos^2\psi d\psi}{\sqrt{1-e^2\sin^2\psi}^{\,3}} \; , \quad \underline{D} = \int_0^{\pi/2} \frac{\sin^2\psi d\psi}{\sqrt{1-e^2\sin^2\psi}} \; , \quad \underline{B} = \int_0^{\pi/2} \frac{\cos^2\psi d\psi}{\sqrt{1-e^2\sin^2\psi}} \; , \tag{3.17b}$$

$$\underline{K} = 2\underline{D}-e^2\underline{C} \; ; \quad \underline{E} = (2-e^2)\underline{D}-e^2\underline{C} \; ; \quad \underline{B} = \underline{D}-e^2\underline{C}. \tag{3.17c}$$

The functions $\underline{K}$ and $\underline{E}$ are the complete elliptic integrals of the first and second kind, respectively. The functions $\underline{B}$, $\underline{C}$, $\underline{D}$ do not have a special name. The five functions are tabulated by JAHNKE & EMDE [1], pg. 78, 80, 83, and 82. In Table 1, we give a small table of the values of $\underline{C}$ and $\underline{D}$, taken from JAHNKE & EMDE [1].

Table 1. $\underline{C}$ and $\underline{D}$ as functions of $g = \sqrt{1-e^2}$.

g	↓0	0.1	0.2	0.3	0.4	0.5
$\underline{C}$	-2+log 4/g	1.7351	1.1239	0.8107	0.6171	0.4863
$\underline{D}$	-1+log 4/g	2.7067	2.0475	1.6827	1.4388	1.2606
g	0.6	0.7	0.8	0.9	1.0	
$\underline{C}$	0.3929	0.3235	0.27060	0.22925	0.19635 $= \frac{\pi}{16}$	
$\underline{D}$	1.1234	1.0138	0.9241	0.8491	0.7854 $= \frac{\pi}{4}$	

It is well-known that the complete elliptic integrals of the type we encountered can be expressed in two independent elliptic integrals. We will list the reduction to $\underline{K}$ and $\underline{E}$, because these functions are widely tabulated. We also give the reduction to $\underline{C}$ and $\underline{D}$, which are tabulated in JAHNKE & EMDE [1], because in our short list of elliptic integrals the coefficients of $\underline{D}$ and $\underline{C}$ do not contain the excentricity $|e|$ in the denominator, while $g^2 = 1-e^2$

occurs in the denominator only twice.

The reduction is accomplished by regarding $\underline{K}$, $\underline{E}$, $\underline{C}$, and $\underline{D}$, and $J(d,i,j,|e|)$ as hypergeometric functions $F(a,b;c;e^2)$ in the following manner. According to ERDÉLYI et al. [1], Vol. 1, pg. 115, eq. 2.12 (7)

$$\left.\begin{aligned} F(a,b;c;z) &= \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\pi/2} \frac{(\cos\psi)^{2c-2b-1}\,(\sin\psi)^{2b-1}d\psi}{(1-z\,\sin^2\psi)^a} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n \quad , \text{ when } |z|<1. \end{aligned}\right\} \quad (3.18a)$$

We set $z=e^2$, $a=d+\frac{1}{2}$, $b=j+\frac{1}{2}$, $c=i+j+1$ in (3.18a), and from this and (3.13) it follows that

$$\left.\begin{aligned} J(d,i,j,|e|) &= \frac{(-1)^{i+j}}{2(d+1)!} \frac{\Gamma(\frac{1}{2}-d+i+j)}{\Gamma(\frac{1}{2}-d)} \int_0^{\pi/2} \frac{(\cos^2\psi)^i(\sin^2\psi)^j d\psi}{(1-e^2\sin^2\psi)^{d+\frac{1}{2}}} \\ &= \frac{(-1)^{i+j}}{4(d+1)!} \frac{\Gamma(\frac{1}{2}-d+i+j)}{\Gamma(\frac{1}{2}-d)} \frac{\Gamma(j+\frac{1}{2})\Gamma(i+\frac{1}{2})}{\Gamma(i+j+1)} F(d+\tfrac{1}{2},\ j+\tfrac{1}{2};\ i+j+1;\ e^2). \end{aligned}\right\} \quad (3.18b)$$

Further we have from (3.17) and (3.18a) that

$$\left.\begin{aligned} &\underline{K} = \frac{\pi}{2} F(\tfrac{1}{2},\tfrac{1}{2};1;e^2); \qquad \underline{E} = \frac{\pi}{2} F(-\tfrac{1}{2},\tfrac{1}{2};1;e^2);\ \underline{B} = \frac{\pi}{4} F(\tfrac{1}{2},\tfrac{1}{2};2;e^2); \\ &\underline{C} = \frac{\pi}{16} F(3/2,3/2;3;e^2);\ \underline{D} = \frac{\pi}{4} F(\tfrac{1}{2},3/2;2;e^2). \end{aligned}\right\} \quad (3.19)$$

The reduction itself is accomplished by repeatedly applying the 15 relations of GAUSS which connect $F(a,b;c;z)$ with any two of the 6 functions $F(a\pm1,b;c;z)$, $F(a,b\pm1;c;z)$, $F(a,b;c\pm1;z)$. These relations can be found, for instance, in ERDÉLYI et al. [1], Vol. 1, par. 2.8, pg. 103-104, eq. (31)/(45). We shall give the result of this reduction without proof. Since according to (3.11) and (3.14)

$$I(d,i,j,e) = 2(d+1)J(d,i,j,e) = 2(d+1)J(d,j,i,-e), \qquad (3.20)$$

we give only $J(d,i,j,|e|)$.

$$\left.\begin{aligned}
J(-1,0,0,|e|) &= \tfrac{1}{2}(2-e^2)\underline{D} - \tfrac{1}{2}e^2\underline{C} &&= \tfrac{1}{2}\,\underline{E},\\
J(-1,0,1,|e|) &= -\tfrac{1}{4}(3-2e^2)\underline{D} + \tfrac{1}{4}e^2\underline{C} &&= -\frac{1-e^2}{4e^2}\,\underline{K} + \frac{1-2e^2}{4e^2}\,\underline{E},\\
J(-1,1,0,|e|) &= -\tfrac{1}{4}(3-e^2)\underline{D} + \tfrac{1}{2}e^2\underline{C} &&= \frac{1-e^2}{4e^2}\,\underline{K} - \frac{1+e^2}{4e^2}\,\underline{E},\\
J(-1,0,2,|e|) &= \frac{1}{8}(11-8e^2)\underline{D} + \frac{1}{8}(1-4e^2)\underline{C} &&= \frac{1+e^2-2e^4}{4e^4}\,\underline{K} - \frac{2+3e^2-8e^4}{8e^4}\,\underline{E},\\
J(-1,1,1,|e|) &= \tfrac{1}{4}(2-e^2)\underline{D} - \frac{1}{8}(1+e^2)\underline{C} &&= -\frac{2-3e^2+e^4}{8e^4}\,\underline{K} + \frac{1-e^2+e^4}{4e^4}\,\underline{E},\\
J(-1,2,0,|e|) &= \frac{1}{8}(11-3e^2)\underline{D} + \frac{1}{8}(1-9e^2)\underline{C} &&= \frac{1-4e^2+3e^4}{4e^4}\,\underline{K} - \frac{2-7e^2-3e^4}{8e^4}\,\underline{E},\\[1ex]
J(0,0,0,|e|) &= \underline{D} - \tfrac{1}{2}e^2\underline{C} &&= \tfrac{1}{2}\,\underline{K},\\
J(0,0,1,|e|) &= -\tfrac{1}{4}\,\underline{D} &&= -\frac{1}{4e^2}\,\underline{K} + \frac{1}{4e^2}\,\underline{E},\\
J(0,1,0,|e|) &= -\tfrac{1}{4}\,\underline{D} + \tfrac{1}{4}e^2\underline{C} &&= \frac{1-e^2}{4e^2}\,\underline{K} - \frac{1}{4e^2}\,\underline{E},\\
J(0,0,2,|e|) &= \tfrac{1}{4}\,\underline{D} + \frac{1}{8}\,\underline{C} &&= \frac{2+e^2}{8e^4}\,\underline{K} - \frac{1+e^2}{4e^4}\,\underline{E}\\
J(0,1,1,|e|) &= \frac{1}{8}\,\underline{D} - \frac{1}{8}\,\underline{C} &&= -\frac{1-e^2}{4e^4}\,\underline{K} + \frac{2-e^2}{8e^4}\,\underline{E},\\
J(0,2,0,|e|) &= \tfrac{1}{4}\,\underline{D} + \frac{1}{8}(1-3e^2)\underline{C} &&= \frac{2-5e^2+3e^4}{8e^4}\,\underline{K} - \frac{1-2e^2}{4e^4}\,\underline{E},\\[1ex]
J(1,0,0,|e|) &= \{(2-e^2)\underline{D} - e^2\underline{C}\}/4g^2 &&= \underline{E}/4g^2,\\
J(1,0,1,|e|) &= \frac{\underline{D}}{8(1-e^2)} - \frac{e^2\underline{C}}{8(1-e^2)} &&= -\frac{1}{8e^2}\,\underline{K} + \frac{1}{8e^2(1-e^2)}\,\underline{E},\\
J(1,1,0,|e|) &= \frac{1}{8}\,\underline{D} &&= \frac{1}{8e^2}\,\underline{K} - \frac{1}{8e^2}\,\underline{E}.
\end{aligned}\right\} \tag{3.21}$$

We can form the following sets of load-displacement equations from the elliptic integrals (3.21):

$X=Y=Z \to \infty$ on edge; w, and (u,v) for $\sigma = 0$: the 2nd degree ($M=2$),
(u,v) for $\sigma \neq 0$: the 1st degree ($M=1$),
$X=Y=Z = 0$ on edge; w, and (u,v) for $\sigma = 0$: the 1st degree ($M=1$),
(u,v) for $\sigma \neq 0$: the 0th degree ($M=0$).

The E's and F's which are needed for those equations are:

$$
\left.\begin{aligned}
\tfrac{1}{2}E^{0;00}_{00} &= F^{0;00}_{00} = s(\underline{D}-\tfrac{1}{2}e^2\underline{C}) = \tfrac{1}{2}s\underline{K} \\
F^{0;00}_{20} &= -s^{-1}(\underline{D}-e^2\underline{C}) = -s^{-1}\underline{B} \\
F^{0;00}_{02} &= -s^{-1}g^2\underline{D} \\
\tfrac{1}{2}E^{0;10}_{10} &= F^{0;10}_{10} = \tfrac{1}{2}s(\underline{D}-e^2\underline{C}) = \tfrac{1}{2}s\underline{B} \\
F^{0;10}_{30} &= -s^{-1}\{2\underline{D}+(1-3e^2)\underline{C}\} = s^{-1}(\underline{D}-3\underline{B}-\underline{C}) \\
F^{0;10}_{12} &= -s^{-1}g^2(\underline{D}-\underline{C}) \\
\tfrac{1}{2}E^{0;01}_{01} &= F^{0;01}_{01} = \tfrac{1}{2}s\underline{D} \\
F^{0;01}_{21} &= -s^{-1}(\underline{D}-\underline{C}) \\
F^{0;01}_{03} &= -s^{-1}g^2(2\underline{D}+\underline{C}) \\
\tfrac{1}{4}E^{0;20}_{00} &= F^{0;20}_{00} = \tfrac{1}{8}s^3\underline{D} \\
\tfrac{1}{2}E^{0;20}_{20} &= F^{0;20}_{20} = \tfrac{1}{2}s\{\underline{D}+(1-2e^2)\underline{C}\} = \tfrac{1}{2}s(2\underline{B}-\underline{D}+\underline{C}) \\
\tfrac{1}{2}E^{0;20}_{02} &= F^{0;20}_{02} = -\tfrac{1}{2}sg^2\underline{C} \\
\tfrac{1}{2}E^{0;11}_{11} &= F^{0;11}_{11} = \tfrac{1}{2}s(\underline{D}-\underline{C}) \\
\tfrac{1}{4}E^{0;02}_{00} &= F^{0;02}_{00} = \frac{s^3}{8g^2}(\underline{D}-e^2\underline{C}) = \frac{s^3}{8g^2}\underline{B} \\
\tfrac{1}{2}E^{0;02}_{20} &= F^{0;02}_{20} = -\tfrac{1}{2}s\underline{C} \\
\tfrac{1}{2}E^{0;02}_{02} &= F^{0;02}_{02} = \tfrac{1}{2}s(\underline{D}+\underline{C}) \\
\tfrac{1}{4}E^{1;00}_{00} &= F^{1;00}_{00} = \frac{s^3}{8g^2}\{(2-e^2)\underline{D}-e^2\underline{C}\} = \frac{s^3}{8g^2}\underline{E} \\
\tfrac{1}{2}E^{1;00}_{20} &= F^{1;00}_{20} = \tfrac{1}{2}s\underline{D} \\
\tfrac{1}{2}E^{1;00}_{02} &= F^{1;00}_{02} = \tfrac{1}{2}s(\underline{D}-e^2\underline{C}) = \tfrac{1}{2}s\underline{B} \\
F^{1;00}_{40} &= -s^{-1}(\underline{D}-\underline{C}) \\
F^{1;00}_{22} &= -g^2s^{-1}\underline{C} \\
F^{1;00}_{04} &= -g^2s^{-1}(\underline{D}-\underline{C}) \\
\tfrac{1}{4}E^{1;10}_{10} &= F^{1;10}_{10} = -\frac{s^3}{8}\underline{D} \\
\tfrac{1}{2}E^{1;10}_{30} &= F^{1;10}_{30} = \tfrac{1}{2}s(\underline{D}-\underline{C}) \\
\tfrac{1}{2}E^{1;10}_{12} &= F^{1;10}_{12} = \tfrac{1}{2}g^2s\underline{C} \\
\tfrac{1}{4}E^{1;01}_{01} &= F^{1;01}_{01} = -\frac{s^3}{8g^2}(\underline{D}-e^2\underline{C}) = -\frac{s^3}{8g^2}\underline{B} \\
\tfrac{1}{2}E^{1;01}_{21} &= F^{1;01}_{21} = \tfrac{1}{2}s\underline{C} \\
\tfrac{1}{2}E^{1;01}_{03} &= F^{1;01}_{03} = \tfrac{1}{2}s(\underline{D}-\underline{C})
\end{aligned}\right\}\quad e \geq 0,\ s = a \tag{3.22}
$$

3.21. The case of infinite surface traction at the edge of the contact area.

In 3.211 we shall treat a normal problem, and in 3.212 a tangential problem in which the traction becomes infinite at the edge of the contact area. So the building blocks of the coefficients of the load-displacement equations are the $E^{h;pq}_{mn}$, see (2.73), (3.13) and (3.14):

$$\left.\begin{aligned} &E^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(e) = \\ &=(\tfrac{1}{2})_h(-2)^{\varepsilon+\omega}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\frac{4^{k+\ell}(2m+\varepsilon)!(2n+\omega)!s^{2d+1}}{(m-k)!(n-\ell)!(2k+\varepsilon)!(2\ell+\omega)!}I(d,k+p+\varepsilon,\ell+q+\omega,e), \\ &d = h+p+q-m-n;\ I(d,i,j,e) = 2(d+1)J(d,i,j,e). \end{aligned}\right\} \quad (3.23)$$

The equations themselves are given in (2.56).

3.211. A normal problem: a rigid, flat elliptical die pressed into a half-space.

A rigid, flat die of elliptical circumference with semi-axes a and b, $s = a \le b$, is pressed into the elastic half-space $z \ge 0$ with a normal force N, with the action line along $x=x_0$, $y=y_0$. The force is applied so, that contact takes place over the whole of the base of the die. Friction is assumed to be absent. This problem was treated by DOVNOROVICH [1] with the aid of the load-displacement equations.

After deformation, the equation of the base of the die is

$$w = z = c_{00} + c_{10}x + c_{01}y; \quad (3.24)$$

the vertical displacement difference w is clearly equal to $w^+(x,y,0)$ since the die is perfectly rigid, and that in turn is clearly given by (3.24). The constants c_{00}, c_{10}, and c_{01} follow from the demand that the total force and moment exerted by the half-space on the die is in equilibrium with the applied load. We have for the normal pressure distribution on the half-space:

$$\left.\begin{aligned} &Z = G\sqrt{1-(x/a)^2-(y/b)^2}^{\,-1}\ (f_{00}+f_{10}x + f_{01}y), \\ &G = 2G^+,\ \sigma = \sigma^+\ ; \end{aligned}\right\} \quad (3.25)$$

it follows from considerations of equilibrium of the die that

$$N = \iint_E Z\, dxdy = 2\pi abG\, f_{00},\quad x_0 N = \iint_E x\, Z\, dxdy = \frac{2}{3}\pi a^3 bG\, f_{10},$$

$$y_0 N = \iint_E y\, Z\, dxdy = \frac{2}{3}\pi ab^3 G\, f_{01},$$

or,

$$f_{00} = N/2\pi abG,\quad f_{10} = \frac{3}{2} Nx_0/\pi a^3 bG,\quad f_{01} = \frac{3}{2} Ny_0/\pi ab^3 G. \qquad (3.26)$$

The condition that contact must take place over the whole of the base of the die is equivalent to the condition that the normal pressure is everywhere positive, that is, according to (3.25) and (3.26), that

$$f_{00} + f_{10}x + f_{01}y = \frac{N}{2\pi abG}\left(1 + \frac{3xx_0}{a^2} + \frac{3yy_0}{b^2}\right) \geq 0, \qquad (3.27a)$$

which after some calculation leads to the condition

$$\frac{x_0^{\,2}}{(\frac{1}{3}a)^2} + \frac{y_0^{\,2}}{(\frac{1}{3}b)^2} \leq 1, \qquad (3.27b)$$

from which we see that (x_0, y_0) must lie inside the ellipse which is concentric, similar, and similarly oriented with E, but the axes of which are $\frac{1}{3}$ times the axes of E.

The load-displacement equations are, according to (2.56c) and (3.22):

$$\left.\begin{aligned}
c_{00} &= 2(1-\sigma)E^{0;00}_{00}\, f_{00} = \frac{(1-\sigma)N}{\pi bG}(2\underline{D}-e^2\underline{C}) = \frac{(1-\sigma)N}{\pi bG}\underline{K},\\
c_{10} &= 2(1-\sigma)E^{0;10}_{10}\, f_{10} = \frac{3(1-\sigma)Nx_0}{\pi a^2 bG}(\underline{D}-e^2\underline{C}) = \frac{3(1-\sigma)Nx_0}{\pi a^2 bG}\underline{B},\\
c_{01} &= 2(1-\sigma)E^{0;01}_{01}\, f_{01} = \frac{3(1-\sigma)Ny_0}{\pi b^3 G}\underline{D},\\
G &= 2G^+,\quad \sigma = \sigma^+,
\end{aligned}\right\} \qquad (3.28)$$

which is also the solution of the problem.

3.212. A tangential problem: the problem of CATTANEO and MINDLIN without slip.

Two elastic bodies are pressed together by a normal force N, so that a contact area forms between them. According to the HERTZ theory, which we assume to be valid, the contact area E is elliptical with semi-axes a, b ($s=a\leq b$):

$$E = \{x,y: (x/a)^2+(y/b)^2 \le 1\}, \; s = a \le b. \tag{3.29}$$

After this, a tangential force (F_x, F_y) and a torsional couple M_z are applied. Assuming that the HERTZ distribution does not influence the tangential displacement difference and vice versa, it is required to find the tangential displacement (δ_x, δ_y) and the torsion angle β of the upper body with respect to the lower. Slip in the contact area is assumed to be absent. This problem was treated by CATTANEO [1] and MINDLIN [1].

Since we must choose the unstressed state so that the displacement vanishes at infinity, we have in the contact area

$$\left.\begin{aligned} u(x,y)&=u^+(x,y,0)-u^-(x,y,0)=\delta_x-\beta y=a_{00}+a_{01}y, \\ v(x,y)&=v^+(x,y,0)-v^-(x,y,0)=\delta_y+\beta x=b_{00}+b_{10}x. \end{aligned}\right\} \tag{3.30}$$

Therefore, the tangential traction distribution over the contact area has the following form:

$$\left.\begin{aligned} X &= G\sqrt{1-(x/a)^2-(y/b)^2}^{\,-1}\,\left(d_{00}+d_{01}y\right), \\ Y &= G\sqrt{1-(x/a)^2-(y/b)^2}^{\,-1}\,\left(e_{00}+e_{10}x\right) \end{aligned}\right\} \tag{3.31a}$$

so that

$$F_x=2\pi abGd_{00},\; F_y=2\pi abGe_{00},\; M_z=\iint_E (xY-yX)\,dxdy=\frac{2}{3}\pi abG(a^2e_{10}-b^2d_{01}) \tag{3.31b}$$

The load-displacement equations are:

$$\left.\begin{aligned} a_{00} &= \delta_x = 2\left(E^{0;00}_{00} - \sigma E^{1;00}_{20}\right)d_{00}, \\ b_{00} &= \delta_y = 2\left(E^{0;00}_{00} - \sigma E^{1;00}_{02}\right)e_{00}, \\ a_{01} &= -\beta = 2\left(E^{0;01}_{01} - \sigma E^{1;01}_{21}\right)d_{01}-2\sigma E^{1;10}_{12}\,e_{10}, \\ b_{10} &= \beta = 2\left(E^{0;10}_{10} - \sigma E^{1;10}_{12}\right)e_{01}-2\sigma E^{1;01}_{21}\,d_{01}. \end{aligned}\right\} \tag{3.32}$$

Now, $e \geq 0$, so that according to (3.22),

$$\left.\begin{aligned} &E^{0;00}_{00} = a\underline{K},\; E^{1;00}_{20} = a\underline{D},\; E^{1;00}_{02} = a\underline{B},\; E^{0;01}_{01} = a\underline{D}, \\ &E^{1;01}_{21} = a\underline{C},\; E^{1;10}_{12} = ag^2\underline{C},\; E^{0;10}_{10} = a\underline{B}. \end{aligned}\right\} \tag{3.33}$$

From (3.17c), (3.31b), (3.32), and (3.33) we can solve δ_x, δ_y and β:

$$\delta_x = \frac{(\underline{K}-\sigma\underline{D})F_x}{\pi bG}, \quad \delta_y = \frac{(\underline{K}-\sigma\underline{B})F_y}{\pi bG}, \quad \beta = \frac{3M_z(\underline{B}\ \underline{D}-\sigma\underline{E}\ \underline{C})}{\pi b^3 G(\underline{E}-4\sigma g^2\underline{C})}. \tag{3.34}$$

3.22. The case of zero surface traction at the edge of the contact area.

In 3.221 we shall treat the HERTZ problem, and in 3.222 the problem of CATTANEO and MINDLIN with slip, but without twist. The HERTZ problem is treated in some detail, since its results are frequently used in the present work. We also give a numerical table.

The building blocks of the coefficients of the load-displacement equations are the $F^{h;pq}_{mn}$ of (3.12):

$$\left. \begin{aligned} F^{h;2p+\varepsilon,2q+\omega}_{2m+\varepsilon,2n+\omega}(e) &= (\tfrac{1}{2})_h(-2)^{\varepsilon+\omega} \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{4^{k+\ell}(2m+\varepsilon)!(2n+\omega)!s^{2d+1}}{(m-k)!(n-\ell)!(2k+\varepsilon)!(2\ell+\omega)!} \times \\ &\times J(d,k+p+\varepsilon,\ell+q+\omega,e), \quad d = h+p+q-m-n \geq -1. \end{aligned} \right\} \tag{3.35}$$

The equations themselves are given by (3.5).

3.221. A normal problem: the problem of HERTZ.

Two elastic bodies are pressed together by a normal force N, so that a contact area forms between them. Assuming that friction is absent, and that for the boundary conditions the bodies may be approximated by elliptic paraboloids, find the contact area, the pressure distribution over the contact area, and the depth of penetration of the bodies.

The most important case in which we shall use the HERTZ problem is that of two bodies of revolution which are steadily rolling over each other. In that case the parallel circles of both bodies are approximately parallel. We shall confine ourselves to that case. The axes of the paraboloids then coincide. The elasticity problem remains the same when the axes of the paraboloids are not parallel, but the boundary conditions require a little more algebra, which is given, for instance in LOVE [1] pg. 193-194. We shall give the results of this analysis only.

We must choose the unstressed state so, that the displacement and the stresses vanish at infinity; in such an unstressed state, the bodies intersect. Let the principal radii of curvature of the bodies be given by $R_x^{\pm}$, $R_y^{\pm}$, where a + refers to the lower body, and a - to the upper body. We count them positive if the centre of curvature in question lies inside the half-space under consideration. The equation of the surface of the bodies near the contact area is

$$z^{\mp} = \mp \frac{x^2}{2R_x^{\mp}} \mp \frac{y^2}{2R_y^{\mp}} + \alpha^{\mp}, \quad \left.\begin{array}{l}\text{upper sign: upper half-space,}\\ \text{lower sign: lower half-space.}\end{array}\right\} \quad (3.36)$$

In the contact area, we have $w^+(x,y,0)+z^+-w^-(x,y,0)-z^- = 0$, that is,

$$w(x,y) = z^-(x,y)-z^+(x,y) = -Ax^2-By^2+\alpha, \quad (3.37)$$

with

$$\alpha = \alpha^- - \alpha^+,$$

$$\left.\begin{array}{l} A = \frac{1}{2}\left(\frac{1}{R_x^+} + \frac{1}{R_x^-}\right), \\ B = \frac{1}{2}\left(\frac{1}{R_y^+} + \frac{1}{R_y^-}\right), \end{array}\right\} \text{ if the axes of the paraboloids are (nearly) parallel,}$$

$$\frac{1}{\rho} = \frac{1}{2}(A+B) = \frac{1}{4}\left(\frac{1}{R_1^+} + \frac{1}{R_1^-} + \frac{1}{R_2^+} + \frac{1}{R_2^-}\right),$$

ρ: characteristic length of the bodies;

$R_{1,2}^{\pm}$: principal radii of curvature of lower (+) and upper (-) body, taken positive when the corresponding centre of curvature lies inside the half-space under consideration; (3.38)

$$4(A-B)^2 = \left(\frac{1}{R_1^+} - \frac{1}{R_2^-}\right)^2 + \left(\frac{1}{R_2^+} - \frac{1}{R_1^-}\right)^2 + 2\left(\frac{1}{R_1^+} - \frac{1}{R_2^-}\right)\left(\frac{1}{R_2^+} - \frac{1}{R_1^-}\right)\cos 2\omega,$$

ω: angle between the plane of R_1^+ and the plane of R_1^-, in case the axes of the paraboloids are not parallel.

This means that

$$c_{00} = \alpha,\ c_{20} = -A,\ c_{02} = -B. \quad (3.39)$$

We propose the hypothesis that the contact area is elliptic with semi-axes a and b,

$$E = \{x,y: (x/a)^2+(y/b)^2 \leq 1\}. \quad (3.40)$$

We take the normal stress in the form

$$Z = G\, f_{00} \sqrt{1-(x/a)^2-(y/b)^2}, \tag{3.41}$$

where G is the combined modulus of rigidity. We will also need the combined POISSON's ratio σ. They are given by (2.10), which we repeat here:

$$\frac{1}{G} = \tfrac{1}{2}\left(\frac{1}{G^+} + \frac{1}{G^-} \right), \quad \frac{\sigma}{G} = \tfrac{1}{2}\left(\frac{\sigma^+}{G^+} + \frac{\sigma^-}{G^-} \right). \tag{3.42}$$

The total normal force can be found from (3.41) by integration:

$$N = \iint_E Z \; dxdy = \frac{2}{3}\, \pi abGf_{00}, \quad f_{00} = \frac{3N}{2\pi abG}. \tag{3.43}$$

The load-displacement equations are

$$\left.\begin{aligned} \alpha = c_{00} &= 2(1-\sigma)F^{0;00}_{\ \ 00}\, f_{00}, \\ -A = c_{20} &= (1-\sigma)F^{0;00}_{\ \ 20}\, f_{00}, \\ -B = c_{02} &= (1-\sigma)F^{0;00}_{\ \ 02}\, f_{00}; \end{aligned}\right\} \tag{3.44}$$

according to (3.22),

$$\left.\begin{aligned} &F^{0;00}_{\ \ 00}(|e|) = F^{0;00}_{\ \ 00}(-|e|) = \tfrac{1}{2}\, s\, \underline{K}, \\ &F^{0;00}_{\ \ 20}(|e|) = F^{0;00}_{\ \ 02}(-|e|) = -(\underline{D}-e^2\underline{C})/s = -\underline{B}/s, \\ &F^{0;00}_{\ \ 02}(|e|) = F^{0;00}_{\ \ 20}(-|e|) = -(1-e^2)\underline{D}/s = -g^2\underline{D}/s. \\ &\text{s: minor semi-axis of contact ellipse.} \end{aligned}\right\} \tag{3.45}$$

So we obtain from (3.43), (3.44), and (3.45):

$$\left.\begin{aligned} \alpha = \frac{3N(1-\sigma)s\underline{K}}{2\pi abG}, \quad A(|e|) = B(-|e|) &= \frac{3N(1-\sigma)(\underline{D}-e^2\underline{C})}{2\pi absG} = \frac{3N(1-\sigma)\underline{B}}{2\pi absG}, \\ B(|e|) = B(-|e|) &= \frac{3N(1-\sigma)(1-e^2)\underline{D}}{2\pi absG} = \frac{3N(1-\sigma)g^2\underline{D}}{2\pi absG}. \end{aligned}\right\} \tag{3.46}$$

Since $\underline{D} > \underline{C}$, see sec. 3.2, Table 1, it follows that $A(|e|)=B(-|e|)\geq B(|e|)=A(-|e|)$, so that we have:

$$\left.\begin{aligned} A = \tfrac{1}{2}\left(\frac{1}{R^+_x} + \frac{1}{R^-_x} \right) \geq B = \tfrac{1}{2}\left(\frac{1}{R^+_y} + \frac{1}{R^-_y} \right) &\Longrightarrow e \geq 0,\ a \leq b, \\ A = \tfrac{1}{2}\left(\frac{1}{R^+_x} + \frac{1}{R^-_x} \right) \leq B = \tfrac{1}{2}\left(\frac{1}{R^+_y} + \frac{1}{R^-_y} \right) &\Longrightarrow e \leq 0,\ b \leq a. \end{aligned}\right\} \tag{3.47}$$

In order to find the excentricity of the contact ellipse, we set with

$$\cos\tau = \frac{|A-B|}{A+B} = \tfrac{1}{2}\rho|A-B| = \frac{|1/R_x^+ + 1/R_x^- - 1/R_y^+ - 1/R_y^-|}{1/R_x^+ + 1/R_x^- + 1/R_y^+ + 1/R_y^-}, \qquad (3.48a)$$

and it follows from this and (3.46) and (3.17c) that

$$\cos\tau = \frac{e^2(\underline{D}-\underline{C})}{\underline{E}}. \qquad (3.48b)$$

$\underline{E}$, $|e|$ and g are tabulated as functions of τ in Table 2. This table is taken from LOVE [1], p. 197, and from JAHNKE & EMDE [1], p. 78 and

Table 2. $|e|$, g, $\underline{E}$, $\underline{K}$ as functions of τ.

τ	90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
$g=s/\ell$	1.00	0.79	0.62	0.47	0.36	0.26	0.18	0.10	0.05	0.00
$\|e\|$	0.00	0.61	0.78	0.88	0.93	0.96	0.98	0.99	0.999	1.00
$\underline{K}$	1.57	1.76	1.97	2.21	2.46	2.75	3.14	3.71	4.40	∞
$\underline{E}$	1.57	1.41	1.29	1.19	1.13	1.08	1.04	1.02	1.01	1.00

80. We see from (3.48) that the shape of the contact ellipse depends only on A and B, and not on the applied load N or the elastic properties of the bodies. The size of the contact area does depend on the load, as follows:

$$A+B = \frac{2}{\rho} = \frac{3N(1-\sigma)\underline{E}}{2\pi abs G} = \frac{3N(1-\sigma)\underline{E}}{2\pi G c^3\sqrt{g}}, \quad c = \sqrt{ab}, \qquad (3.49)$$

or

$$3N(1-\sigma)\rho\underline{E} = 4\pi c^3 G\sqrt{g}, \quad c = \sqrt{ab}. \qquad (3.50)$$

A frequently-used quantity is f_{00}. It is

$$f_{00} = \frac{3N}{2\pi ab G} = \frac{2c\sqrt{g}}{(1-\sigma)\underline{E}\rho} = \frac{2}{(1-\sigma)\underline{E}}\;\frac{s}{\rho}. \qquad (3.51)$$

Finally we determine the penetration α of the bodies according to (3.44), (3.46), (3.51)

$$\alpha = (1-\sigma)\underline{K}\, f_{00}\, s = \frac{2\, s^2\, \underline{K}}{\rho\underline{E}}. \qquad (3.52)$$

3.222. A tangential problem: The problem of CATTANEO and MINDLIN with slip, without twist.

Two elastic bodies are pressed together by a normal force N, so that a contact area forms between them. According to the HERTZ theory, which we assume to be applicable, the contact area E is elliptical with semi-axes a and b, $a \leq b$:

$$E = \{x,y:\ (x/a)^2+(y/b)^2\},\ a \leq b. \tag{3.53}$$

After this, a tangential force (F_x,F_y) is applied. Assuming that the HERTZ distribution does not influence the tangential displacement difference, and vice versa, it is required to find the tangential displacement (δ_x,δ_y) of the upper body with respect to the lower. This problem was treated by MINDLIN [1] and CATTANEO [1].

If the tangential force is below its maximal value as predicted by COULOMB's law,

$$|(F_x,F_y)|<\mu N,\quad \mu\text{: coefficient of friction} \tag{3.54}$$

the contact area is split up into a region of adhesion E_h in which there is no relative movement of the particles in contact as a consequence of the tangential force, and a region of slip E_g where the tangential traction has reached the COULOMB value $|(X,Y)|=\mu Z$. The boundary conditions in E_h are the same as those of 3.312, with $\beta=0$:

$$\left.\begin{aligned} u(x,y) &= u^+(x,y,0)-u^-(x,y,0) = \delta_x\ , \\ v(x,y) &= v^+(x,y,0)-v^-(x,y,0) = \delta_y\ , \end{aligned}\right\}\ \text{in } E_h\ . \tag{3.55}$$

The boundary conditions in E_g are, that the tangential traction is equal to the COULOMB value, and that the local slip takes place in the direction of the local tangential traction:

$$\left.\begin{aligned} &|(X,Y)|=\mu Z = G\mu\ f_{00}\ \sqrt{1-(x/a)^2-(y/b)^2},\ f_{00} = \frac{3N}{2\pi abG}\ , && (3.56a)\\ &\text{slip in direction of tangential traction.} && (3.56b) \end{aligned}\right\}\ \text{in } E_g$$

In the analysis of CATTANEO and MINDLIN, which we will give here with the aid of the load-displacement equations, boundary conditions (3.55) and (3.56a) are met completely; (3.56b) is satisfied only approximately, for it is assumed that (X,Y) is in the same sense as (F_x,F_y), rather than in the same sense as the slip. The solution is found by a device which was already used by CARTER [1] in his treatment of the problem of the rolling contact with creepage of parallel cylinders. This device consists of assuming that the stress distribution is that which obtains when complete sliding takes place, (X',Y'), from which is substracted a stress distribution (X",Y") over the adhesion area alone, and which is similar to the stress distribution of complete sliding. As a consequence (3.56a) is met automatically and, (this hypothesis was advanced by CATTANEO and MINDLIN), the area of adhesion will be bounded by an ellipse. We will show that the ellipse is similar to the contact ellipse, concentric with it, and similarly oriented. We denote the semi-axes of the area of adhesion by a",b", and we will prove the statement just made by showing that the boundary conditions (3.55) can be met.

Denoting by (u',v') the displacement differences due to the stress distribution (X',Y') of complete sliding, and by (u",v") those due to the stress distribution (X",Y") over the adhesion area alone, we have

$$\left.\begin{aligned}
(X',Y') &= \mu Z\,\frac{(F_x,F_y)}{F} = \frac{\mu G f_{00}}{F}\,(F_x,F_y)\sqrt{1-(x/a)^2-(y/b)^2} && \text{in } E,\\
&= 0 && \text{outside } E,\\
(X'',Y'') &= \mu G f''_{00}\,\frac{(F_x,F_y)}{F}\,\sqrt{1-(x/a'')^2-(y/b'')^2} && \text{in } E_h,\\
&= 0 && \text{outside } E_h,\\
(X,Y) &= (X',Y')-(X'',Y''); \quad F = |(F_x,F_y)|,
\end{aligned}\right\} \qquad (3.57)$$

and

$$(u',v') = (a_{00},b_{00})+(a_{20},b_{20})x^2+(a_{11},b_{11})xy+(a_{02},b_{02})y^2 \quad \text{in } E, \quad (3.58a)$$

$$(u'',v'') = (a''_{00},b''_{00})+(a''_{20},b''_{20})x^2+(a''_{11},b''_{11})xy+(a''_{02},b''_{02})y^2 \quad \text{in } E_h, \quad (3.58b)$$

$$(u,v) = (u'-u'',\ v'-v'') = (\delta_x,\delta_y) \quad \text{in } E_h, \quad (3.58c)$$

where, according to the load-displacement equations (3.6),

$$\left.\begin{aligned}
a_{00} &= 2\left(F^{0;00}_{00} - \sigma F^{1;00}_{20}\right)\\
a_{20} &= \left(F^{0;00}_{20} - \sigma F^{1;00}_{40}\right)\\
b_{11} &= -2\sigma F^{1;00}_{22}\\
a_{02} &= \left(F^{0;00}_{02} - \sigma F^{1;00}_{22}\right)
\end{aligned}\right\}\mu f_{00}\frac{F_x}{F}, \qquad
\left.\begin{aligned}
a''_{00} &= 2\left(F''^{0;00}_{00} - \sigma F''^{1;00}_{20}\right)\\
a''_{20} &= \left(F''^{0;00}_{20} - \sigma F''^{1;00}_{40}\right)\\
b''_{11} &= -2\sigma F''^{1;00}_{22}\\
a''_{02} &= \left(F''^{0;00}_{02} - \sigma F''^{1;00}_{22}\right)
\end{aligned}\right\}\mu f''_{00}\frac{F_x}{F}, \qquad (3.59a)$$

$$\left.\begin{aligned}
b_{00} &= 2\left(F^{0;00}_{00} - \sigma F^{1;00}_{02}\right) \\
b_{20} &= \left(F^{0;00}_{20} - \sigma F^{1;00}_{22}\right) \\
a_{11} &= -2\sigma F^{1;00}_{22} \\
b_{02} &= \left(F^{0;00}_{02} - \sigma F^{1;00}_{04}\right)
\end{aligned}\right\} \mu f_{00} \frac{F_y}{F}, \quad
\left.\begin{aligned}
b''_{00} &= 2\left(\overset{''}{F}{}^{0;00}_{00} - \sigma \overset{''}{F}{}^{1;00}_{02}\right) \\
b''_{20} &= \left(\overset{''}{F}{}^{0;00}_{20} - \sigma \overset{''}{F}{}^{1;00}_{22}\right) \\
a''_{11} &= -2\, \overset{''}{F}{}^{1;00}_{22} \\
b''_{02} &= \left(\overset{''}{F}{}^{0;00}_{02} - \sigma \overset{''}{F}{}^{1;00}_{04}\right)
\end{aligned}\right\} \mu f''_{00} \frac{F_y}{F}.$$

(3.59b)

Here the coefficients $F^{h;pq}_{mn}$ are taken with the minor semi-axis a of the contact area, while the $\overset{''}{F}{}^{h;pq}_{mn}$ are taken with the minor semi-axis a" of the adhesion area.

Now, we see from (3.35) that the coefficients F and $\overset{''}{F}$ of the second degree terms are equal to each other but for a factor $s^{-1}=a^{-1}$ and $(s'')^{-1}=(a'')^{-1}$, since d= -1. So,

$$\overset{''}{F} = F\, a/a'' \text{ in 2nd degree terms.} \tag{3.60}$$

If the second degree terms in (u,v) are to vanish in E_h, as is demanded by (3.58c), we must choose

$$f''_{00} = + \frac{a''}{a} f_{00}. \tag{3.61}$$

If we do so <u>all</u> second degree terms vanish simultaneously.

We are now in a position to express the semi-axes a" in a, with the aid of the prescribed forces F_x and F_y:

$$F_x = \iint_E X'\,dxdy - \iint_{E_h} X''\,dxdy = \iint_E X'\,dxdy - \iint_E \frac{a''b''}{ab}\frac{a''}{a} X'\,dxdy =$$
$$= \left\{1-(a''/a)^3\right\} \mu F_x N/F,$$
$$F_y = \left\{1-(a''/a)^3\right\} \mu F_y N/F,$$

so that

$$\frac{b''}{b} = \frac{a''}{a} = (1-F/\mu N)^{1/3}, \quad F = \sqrt{F_x^2 + F_y^2}. \tag{3.62}$$

As to the zero degree terms, it follows from the fact that d=0, that $\overset{''}{F} = F\, a''/a$, so that

$$a''_{00} = a_{00} \frac{f''_{00}}{f_{00}} \frac{a''}{a} = a_{00}(1-F/\mu N)^{2/3}, \; b''_{00}=b_{00}(1-F/\mu N)^{2/3}. \tag{3.63}$$

According to (3.22),

$$F^{0;00}_{00} = \tfrac{1}{2}\underline{K}a, \quad F^{1;00}_{20} = \tfrac{1}{2}\underline{D}a, \quad F^{1;00}_{02} = \tfrac{1}{2}\underline{B}a, \tag{3.64}$$

and we finally find that

$$\left.\begin{aligned} \delta_x &= \{1-(1-F/\mu N)^{2/3}\}(\underline{K}-\sigma\underline{D})\,\frac{3\mu NF_x}{2\pi bGF}\,, \\ \delta_y &= \{1-(1-F/\mu N)^{2/3}\}(K-\sigma\underline{B})\,\frac{3\mu NF_y}{2\pi bGF}\,. \end{aligned}\right\} \tag{3.65}$$

If we let $F/\mu N$ approach zero, we get again the result (3.34).

It should be observed that for non-vanishing POISSON's ratio σ the boundary condition (3.56b) is met only approximately. In order to see that, we consider the case that $F_y=0$, and that F_x grows to $F_x=\mu N$.

The traction at every instant is then parallel to the x-axis, and the same should hold for the slip. The slip is given by $\left(\frac{\partial[u-\delta_x]}{\partial t}, \frac{\partial[v-\delta_y]}{\partial t}\right)$; its y-component should vanish, that is, $\frac{\partial(v-\delta_y)}{\partial t} = 0$. Since $\delta_y=0$ when $F_y=0$, $\frac{\partial v}{\partial t}$ should vanish at every instant. Accordingly, v should vanish in the final state of complete slip; in that case, $v''=0$, and $v=v'=b_{11}xy$ according to (3.59a), where $b_{11}\neq 0$ when $\sigma\neq 0$. So the slip is not always parallel to the traction. In the case of a circular contact area, the maximum angle between (u,v) and (X,Y) is 9.6° when $\sigma=\frac{1}{2}$, and 4.1° when $\sigma=\frac{1}{4}$. We conjecture from this that the angle between (u,v) and (X,Y) is always small.