Computation and Application of Balanced Model Order Reduction

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- Support: NSF and AFOSR

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Projection Methods: Large Scale Problems

Brief Intro to Model Reduction

Gramian Based Model Reduction: Balanced Reduction

Solving Large Lyapunov Equations: Approximate Balancing
1M vars now possible

Balanced Reduction of Oseen Eqns: Extension to Descriptor System

Domain Decomposition: Couple Linear with Nonlinear Domains

Neural Modeling: Local Reduction ⇒ Many Interactions
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du \]

\[ A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m} \quad n \gg m, p \]

Construct LOW dimensional system

\[ \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \]
\[ \hat{y} = \hat{C}\hat{x} + Du \]

Goal: \( \hat{y} \) should approximate \( y \)

Want the small system response to be the same as the original system response to the same input
## Application Examples

| 1. Passive devices       | • VLSI circuits  
|                          | • Thermal issues |
| 2. Data assimilation     | • North sea forecast  
|                          | • Air quality forecast  
|                          | • Sensor placement  |
| 3. Biological/Molecular systems | • Honeycomb vibrations  
|                                | • MD simulations  
|                                | • Heat capacity  |
| 4. CVD reactor           | • Bifurcations  |
| 5. Mechanical systems:   | • Windscreen vibrations  
|                          | • Buildings  |
| 6. Optimal cooling       | • Steel profile  |
| 7. MEMS: Micro Elec-Mech Systems | • Elf sensor  |
Passive Devices: VLSI circuits

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</table>
**LTI Systems and Model Reduction**

**Time Domain**

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

\[A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}, \ n \gg m, p\]

**Frequency Domain**

\[sx = Ax + Bu \\
y = Cx\]

**Transfer Function**

\[H(s) \equiv C(sI - A)^{-1}B, \quad y(s) = H(s)u(s)\]

This is an \(p \times m\) matrix with rational functions as entries.
Construct a new system \( \{\hat{A}, \hat{B}, \hat{C}\} \) with LOW dimension \( k << n \)

\[
\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\
\hat{y} = \hat{C}\hat{x}
\]

**Goal:** Preserve system response

\( \hat{y} \) should approximate \( y \)

Rational Approximation:

\[
y - \hat{y} = (H(s) - \hat{H}(s))u(s)
\]
Approximate $x \in S_V = \text{Range}(V)$ $k$-diml. subspace i.e. Put $x = V\hat{x}$, and then force

$$W^T[V\dot{\hat{x}} - (AV\hat{x} + Bu)] = 0$$

$$\hat{y} = CV\hat{x}$$

If $W^TV = I_k$, then the $k$ dimensional reduced model is

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$$

$$\hat{y} = \hat{C}\hat{x}$$

where $\hat{A} = W^TAV$, $\hat{B} = W^TB$, $\hat{C} = CV$. 
Moment Matching ↔ Krylov Subspace Projection

Based on Lanczos, Arnoldi, Rational Krylov methods

Padé via Lanczos (PVL)
Freund, Feldmann
Bai

Multipoint Rational Interpolation
Grimme
Gallivan, Grimme, Van Dooren
Recent: Optimal $H_2$ approximation via interpolation
Gugercin, Antoulas, Beattie
Gramian Based Model Reduction

Proper Orthogonal Decomposition (POD)
Principal Component Analysis (PCA)

\[
\dot{x}(t) = f(x(t), u(t)), \quad y = g(x(t), u(t))
\]

The Gramian

\[
P = \int_{0}^{\infty} x(\tau)x(\tau)^T d\tau
\]

Eigenvectors of \(P\)

\[
P = VS^2V^T
\]

Orthogonal Basis

\[
x(t) = VS\mathbf{w}(t)
\]
PCA or POD Reduced Basis

Low Rank Approximation

\[ \mathbf{x} \approx \mathbf{V}_k \hat{\mathbf{x}}_k(t) \]

Galerkin condition – Global Basis

\[ \dot{\hat{\mathbf{x}}}_k = \mathbf{V}_k^T \mathbf{f}(\mathbf{V}_k \hat{\mathbf{x}}_k(t), \mathbf{u}(t)) \]

Global Approximation Error \((\mathcal{H}_2 \text{ bound for LTI})\)

\[ \| \mathbf{x} - \mathbf{V}_k \hat{\mathbf{x}}_k \|_2 \approx \sigma_{k+1} \]

Snapshot Approximation to \(\mathcal{P}\)

\[ \mathcal{P} \approx \frac{1}{m} \sum_{j=1}^{m} \mathbf{x}(t_j) \mathbf{x}(t_j)^T = \mathbf{X} \mathbf{X}^T \]

Truncate SVD: \(\mathbf{X} = \mathbf{V} \mathbf{S} \mathbf{U}^T \approx \mathbf{V}_k \mathbf{S}_k \mathbf{U}^T_k\)
SVD Compression

 Advantage of SVD Compression

\[ k(m + n) \]

v.s.

\[ m \times n \]

Storage

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Image Compression - Feature Detection

original

rank = 10

rank = 30

rank = 50

rank = 50
POD in CFD

Extensive Literature
Karhunen-Loéve, L. Sirovich
Burns, King
Kunisch and Volkwein
Gunzburger
Many, many others

Incorporating Observations – Balancing
Lall, Marsden and Glavaski
K. Willcox and J. Peraire
POD for LTI systems

Impulse Response: \[ \mathcal{H}(t) = C(tI - A)^{-1}B, \quad t \geq 0 \]

Input to State Map: \[ x(t) = e^{At}B \]

Controllability Gramian:

\[ P = \int_{0}^{\infty} x(\tau)x(\tau)^T \, d\tau = \int_{0}^{\infty} e^{A\tau}BB^T e^{A^T\tau} \, d\tau \]

State to Output Map: \[ y(t) = Ce^{At}x(0) \]

Observability Gramian:

\[ Q = \int_{0}^{\infty} e^{A^T\tau}C^TCe^{A\tau} \, d\tau \]
Balanced Reduction (Moore 81)

Lyapunov Equations for system Gramians

\( AP + PA^T + BB^T = 0 \quad A^T Q + QA + C^T C = 0 \)

With \( P = Q = S \) : Want Gramians Diagonal and Equal

States **Difficult to Reach** are also **Difficult to Observe**

Reduced Model \( A_k = W^T_k AV_k \), \( B_k = W^T_k B \), \( C_k = C_k V_k \)

- \( PV_k = W_k S_k \)
- \( QW_k = V_k S_k \)
- Reduced Model Gramians \( P_k = S_k \) and \( Q_k = S_k \).
Why Balanced Truncation?

- Hankel singular values $= \sqrt{\lambda(PQ)}$
- Model reduction $\mathcal{H}_\infty$ error (Glover)

$$\|y - \hat{y}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$

- Extends to MIMO
- Preserves Stability

Key Challenge

- Approximately solve large scale Lyapunov Equations in Low Rank Factored Form
CD Player Frequency Response

Freq–Response CD–Player: $\tau = 0.01$, $n = 120$, $k = 9$
CD Player Frequency Response

Freq–Response CD–Player : $\tau = 0.001$, $n = 120$, $k = 12$

$|G(j\omega)|$

Frequency $\omega$

Original
Reduced
CD Player Frequency Response

Freq–Response CD–Player: $\tau = 1\times10^{-5}$, $n = 120$, $k = 37$
$\sqrt{\lambda(PQ)}$

CD Player - Hankel Singular Values

Hankel Singular Values

$10^{-14}$

$10^{-12}$

$10^{-10}$

$10^{-8}$

$10^{-6}$

$10^{-4}$

$10^{-2}$

$10^{0}$

$10^{2}$

$0$  $20$  $40$  $60$  $80$  $100$  $120$
When are Low Rank Solutions Expected = Eigenvalue Decay in $\mathcal{P}$, $\mathcal{Q}$
**Approximate Balancing**

\[ \mathbf{A} \mathbf{P} + \mathbf{P} \mathbf{A}^T + \mathbf{B} \mathbf{B}^T = 0 \quad \mathbf{A}^T \mathbf{Q} + \mathbf{Q} \mathbf{A} + \mathbf{C}^T \mathbf{C} = 0 \]

- Sparse Case: Iteratively Solve in Low Rank Factored Form,
  \[ \mathbf{P} \approx \mathbf{U}_k \mathbf{U}_k^T, \quad \mathbf{Q} \approx \mathbf{L}_k \mathbf{L}_k^T \]

\[ [\mathbf{X}, \mathbf{S}, \mathbf{Y}] = \text{svd}(\mathbf{U}_k^T \mathbf{L}_k) \]

\[ \mathbf{W}_k = \mathbf{L} \mathbf{Y}_k \mathbf{S}_k^{-1/2} \quad \text{and} \quad \mathbf{V}_k = \mathbf{U} \mathbf{X}_k \mathbf{S}_k^{-1/2}. \]

Now: \[ \mathbf{P} \mathbf{W}_k \approx \mathbf{V}_k \mathbf{S}_k \quad \text{and} \quad \mathbf{Q} \mathbf{V}_k \approx \mathbf{W}_k \mathbf{S}_k \]
Balanced Reduction via Projection

Reduced model of order $k$:

$$A_k = W_k^T A V_k, \quad B_k = W_k^T B, \quad C_k = C V_k.$$ 

$$0 = W_k^T (AP + PA^T + BB^T) W_k = A_k S_k + S_k A_k^T + B_k B_k^T$$

$$0 = V_k^T (A^T Q + QA + C^T C) V_k = A_k^T S_k + S_k A_k + C_k^T C_k$$

Reduced model is balanced and asymptotically stable for every $k$. 

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Low Rank Smith = ADI

Convert to Stein Equation:

\[ \mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \iff \mathcal{P} = \mathbf{A}_\mu\mathcal{P}\mathbf{A}_\mu^T + \mathbf{B}_\mu\mathbf{B}_\mu^T, \]

where

\[ \mathbf{A}_\mu = (\mathbf{A} - \mu\mathbf{I})(\mathbf{A} + \mu\mathbf{I})^{-1}, \quad \mathbf{B}_\mu = \sqrt{2|\mu|}(\mathbf{A} + \mu\mathbf{I})^{-1}\mathbf{B}. \]

Solution:

\[ \mathcal{P} = \sum_{j=0}^{\infty} \mathbf{A}_\mu^j\mathbf{B}_\mu\mathbf{B}_\mu^T(\mathbf{A}_\mu^j)^T = \mathbf{L}\mathbf{L}^T, \]

where \( \mathbf{L} = [\mathbf{B}_\mu, \mathbf{A}_\mu\mathbf{B}_\mu, \mathbf{A}_\mu^2\mathbf{B}_\mu, \ldots] \) Factored Form
Multi-Shift (Modified) Low Rank Smith

LR - Smith: Update Factored Form $\mathcal{P}_m = \mathbf{L}_m \mathbf{L}_m^T$:
(Penzl, Li, White)

\[
\mathbf{L}_{m+1} = [\mathbf{A}_\mu \mathbf{L}_m, \mathbf{B}_\mu]
\]
\[
= [\mathbf{A}_{\mu}^{m+1} \mathbf{B}_\mu, \mathbf{L}_m]
\]

Multi-Shift LR - Smith: (Gugercin, Antoulas, and S.)
Update and Truncate SVD
Re-Order and Aggregate Shift Applications
Much Faster and Far Less Storage

\[
\mathbf{B} \leftarrow \mathbf{A}_\mu \mathbf{B}; \\
[\mathbf{V}, \mathbf{S}, \mathbf{Q}] = \text{svd}([\mathbf{A}_\mu \mathbf{B}, \mathbf{L}_m]); \\
\mathbf{L}_{m+1} \leftarrow \mathbf{V}_k \mathbf{S}_k; \quad (\sigma_{k+1} < tol \cdot \sigma_1)
\]
Approximate Power Method (Hodel)

\[ AP\mathbf{U} + \mathcal{P} A^T \mathbf{U} + BB^T \mathbf{U} = 0 \]
\[ AP\mathbf{U} + \mathcal{P} \mathbf{U} \mathbf{U}^T A^T \mathbf{U} + BB^T \mathbf{U} + \mathcal{P}(\mathbf{I} - \mathbf{U} \mathbf{U}^T)A^T \mathbf{U} = 0 \]

Thus

\[ AP\mathbf{U} + \mathcal{P} \mathbf{U} \mathbf{H}^T + BB^T \mathbf{U} \approx 0 \quad \text{where} \quad \mathbf{H} = \mathbf{U}^T A \mathbf{U} \]

Solving

\[ A\mathbf{Z} + \mathbf{Z} \mathbf{H}^T + BB^T \mathbf{U} = 0 \]

gives approximation to

\[ \mathbf{Z} \approx \mathcal{P} \mathbf{U} \]

Iterate \( \Rightarrow \) Approximate Power Method \( \mathbf{Z}_j \rightarrow \mathbf{US} \) with \( \mathcal{P} \mathbf{U} = \mathbf{US} \)
(also see Vasilyev and White 05)
A Parameter Free Synthesis ($\mathcal{P} \approx \mathbf{U} \mathbf{S}^2 \mathbf{U}^T$)

**Step 1:** Solve the reduced order Lyapunov equation

Solve $\mathbf{H} \hat{\mathbf{P}} + \hat{\mathbf{P}} \mathbf{H}^T + \hat{\mathbf{B}} \hat{\mathbf{B}}^T = 0$.

with $\mathbf{H} = \mathbf{U}_k^T \mathbf{A} \mathbf{U}_k$, $\hat{\mathbf{B}} = \mathbf{U}_k^T \mathbf{B}$.

**Step 2:** (APM step) Solve a projected Sylvester equation

$\mathbf{A} \mathbf{Z} + \mathbf{Z} \mathbf{H}^T + \hat{\mathbf{B}} \hat{\mathbf{B}}^T = 0$.

**Step 3:** Modify $\mathbf{B}$

Update $\mathbf{B} \leftarrow (\mathbf{I} - \mathbf{Z} \hat{\mathbf{P}}^{-1} \mathbf{U}^T) \mathbf{B}$.

**Step 4:** (ADI step) Update factorization and basis $\mathbf{U}_k$

Re-scale $\mathbf{Z} \leftarrow \mathbf{Z} \hat{\mathbf{P}}^{-1/2}$.

Update (and truncate) $[\mathbf{U}, \mathbf{S}] \leftarrow \text{svd}[\mathbf{US}, \mathbf{Z}]$.

$\mathbf{U}_k \leftarrow \mathbf{U}(\cdot, 1 : k)$, basis for dominant subspace.
Automatic Shift Selection - Placement?

SUPG discretization advection-diffusion operator on square grid

\[ k = 32, \ m = 59, \ n = 32 \times 32, \quad \text{Thanks Embree} \]
$\epsilon$-Pseudospectra for $A$ from SUPG, $n=32 \times 32$
Convergence History, Supg, n = 32, N = 1024

Laptop

<table>
<thead>
<tr>
<th>Iter</th>
<th>( \frac{| P_+ - P |}{| P_+ |} )</th>
<th>( B_j )</th>
<th>( \hat{B}_j )</th>
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<tr>
<td>1</td>
<td>2.7e-1</td>
<td>1.6e+0</td>
<td>4.7e+0</td>
</tr>
<tr>
<td>2</td>
<td>7.2e-2</td>
<td>1.6e-1</td>
<td>1.5e+0</td>
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<tr>
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<td>1.1e-2</td>
<td>1.3e-1</td>
</tr>
<tr>
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<td>3.7e-4</td>
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<tr>
<td>5</td>
<td>9.3e-9</td>
<td>1.4e-11</td>
<td>3.4e-7</td>
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</table>

\( P_f \) is rank \( k = 59 \)

\[ \frac{\| P - P_f \|}{\| P \|} = 8.8e - 9 \]

\[ \text{Comptime}(P_f) = 16.2 \text{ secs} \]

\[ \text{Comptime}(P) = 810 \text{ secs} \]
**Convergence History, Supg, n = 800, N = 640,000**

CaamPC

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<tr>
<th>Iter</th>
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<th>$|B_j|$</th>
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<td>6.4e-07</td>
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</table>

$P_f$ is rank $k = 120$  

Comptime($P_f$) = 157 mins = 2.6 hrs
A Descriptor System

\[ E_{11} \frac{d}{dt} v(t) = A_{11} v(t) + A_{12} p(t) + B_1 g(t), \]
\[ 0 = A_{12}^T v(t), \]
\[ v(0) = v_0, \]
\[ y(t) = C_1 v(t) + C_2 p(t) + D g(t). \]

Note \( E \) is singular \[ \text{index 2} \]
Notation

Put

\[ \tilde{E} = \Pi E_{11} \Pi^T, \quad \tilde{A} = \Pi A_{11} \Pi^T, \quad \tilde{B} = \Pi B_1, \quad \tilde{C} = C \Pi^T. \]

With this notation,

\[ \tilde{A} P \tilde{E} + \tilde{E} P \tilde{A}^T + \tilde{B} \tilde{B}^T = 0, \]

\[ \tilde{A}^T Q \tilde{E} + \tilde{E} Q \tilde{A} + \tilde{C}^T \tilde{C} = 0. \]

where

\[ \Pi = I - A_{12} (A_{12}^T E_{11}^{-1} A_{12})^{-1} A_{12}^T E_{11}^{-1} \]

\[ = \Theta_l \Theta_r^T \]

\[ \Pi^T \text{ Projector onto } \text{Null}(A_{12}^T) \]
ADI Derivation Step 1

Begin with

\[ \tilde{A}P \left( \tilde{E} + \bar{\mu} \tilde{A} \right)^T = - \left[ \left( \tilde{E} - \bar{\mu} \tilde{A} \right) P \tilde{A}^T + \tilde{B} \tilde{B}^T \right]. \]

and derive

\[ \left( \tilde{E} + \mu \tilde{A} \right) P \left( \tilde{E} + \bar{\mu} \tilde{A} \right)^T = \left( \tilde{E} - \bar{\mu} \tilde{A} \right) P \left( \tilde{E} - \mu \tilde{A} \right)^T - 2 \text{Re}(\mu) \tilde{B} \tilde{B}^T. \]

Problem: \( \tilde{E} + \mu \tilde{A} \) is *Singular*
Suppose $\Theta_r^T E_{11} \Theta_r + \mu \Theta_r^T A_{11} \Theta_r$ is invertible.

Then the matrix

$$
\left( \tilde{E} + \mu \tilde{A} \right)^I \equiv \Theta_r \left( \Theta_r^T E_{11} \Theta_r + \mu \Theta_r^T A_{11} \Theta_r \right)^{-1} \Theta_r^T
$$

satisfies

$$
\left( \tilde{E} + \mu \tilde{A} \right)^I \left( \tilde{E} + \mu \tilde{A} \right) = \Pi^T
$$

and

$$
\left( \tilde{E} + \mu \tilde{A} \right) \left( \tilde{E} + \mu \tilde{A} \right)^I = \Pi.
$$
Projected Stein Equation

\[ P = \tilde{A}_\mu P \tilde{A}_\mu^* - 2 \text{Re}(\mu) \tilde{B}_\mu \tilde{B}_\mu^*. \]

where

\[ \tilde{A}_\mu \equiv (\tilde{E} + \mu \tilde{A})' (\tilde{E} - \bar{\mu} \tilde{A}), \quad \text{and} \quad \tilde{B}_\mu \equiv (\tilde{E} + \mu \tilde{A})' \tilde{B} \]

Solution:

\[ P = -2 \text{Re}(\mu) \sum_{j=0}^{\infty} \tilde{A}_\mu^j \tilde{B}_\mu \tilde{B}_\mu^* (\tilde{A}_\mu^*)^j. \]

Convergent for stable pencil with Real(\mu) < 0
If $\mathbf{M} = \Pi^T \mathbf{M}$, then the computation

$$Z = \left( \tilde{\mathbf{E}} + \mu \tilde{\mathbf{A}} \right)^I \left( \tilde{\mathbf{E}} - \bar{\mu} \tilde{\mathbf{A}} \right) \mathbf{M}$$

may be accomplished with the following steps.

1. Put $\mathbf{F} = (\mathbf{E}_{11} - \bar{\mu} \mathbf{A}_{11}) \mathbf{M}$.

2. Solve

$$\begin{pmatrix} \mathbf{E}_{11} + \mu \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & 0 \end{pmatrix} \begin{pmatrix} Z \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix}.$$ 

Note that $\mathbf{Z}$ satisfies $\mathbf{Z} = \Pi^T \mathbf{Z}$

Similar result holds for computing $\tilde{\mathbf{B}}_\mu$
Algorithm: Single Shift ADI

1. Solve \[
\begin{pmatrix}
E_{11} + \mu A_{11} & A_{12} \\
A_{12}^T & 0
\end{pmatrix}
\begin{pmatrix}
Z \\
\Lambda
\end{pmatrix} =
\begin{pmatrix}
B \\
0
\end{pmatrix};
\]

2. \( U = Z; \)

3. while ( ’not converged’)
   3.1 \( Z \leftarrow (E_{11} - \bar{\mu} A_{11})Z; \)
   3.2 Solve (in place) \[
\begin{pmatrix}
E_{11} + \mu A_{11} & A_{12} \\
A_{12}^T & 0
\end{pmatrix}
\begin{pmatrix}
Z \\
\Lambda
\end{pmatrix} =
\begin{pmatrix}
Z \\
0
\end{pmatrix};
\]
   3.3 \( U \leftarrow [U, Z]; \)
end

4. \( U \leftarrow \sqrt{2|\text{Re}(\mu)|} U. \)
Derivation Multi-Shift ADI

Easy to see \((P - P_k) = \tilde{A}_\mu^k P \left(\tilde{A}_\mu^*\right)^k\).

Hence

\[
\tilde{A}(P - P_k)\tilde{E} + \tilde{E}(P - P_k)\tilde{A}^* = \tilde{A}\tilde{A}_\mu^k P \left(\tilde{A}_\mu^k\right)^* \tilde{E} + \tilde{E}\tilde{A}_\mu^k P \left(\tilde{A}_\mu^k\right)^* \tilde{A}^*
\]

\[
= \hat{A}_\mu^k \left(\tilde{A}P\tilde{E} + \tilde{E}P\tilde{A}^*\right) \left(\tilde{A}_\mu^k\right)^*.
\]

To get

\[
\tilde{A}(P - P_k)\tilde{E} + \tilde{E}(P - P_k)\tilde{A}^* = -\hat{A}_\mu^k \left(\tilde{B}\tilde{B}^*\right) \left(\tilde{A}_\mu^k\right)^*.
\]

Where

\[
\hat{A}_\mu \equiv \left(\tilde{E} - \bar{\mu} \tilde{A}\right) \left(\tilde{E} + \mu \tilde{A}\right)^I.
\]
Algorithm: Multi-Shift ADI

1. \( U = [ ] \);
2. while ( ’not converged’) 
   for \( i = 1:m \),
   2.1 Solve \( \begin{pmatrix} \mathbf{E}_{11} + \mu_i \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ 0 \end{pmatrix} \); 
   2.2 \( U_0 = \mathbf{Z} \); 
   2.3 for \( j = 1:k-1 \),
   2.3.1 \( \mathbf{Z} \leftarrow (\mathbf{E}_{11} - \bar{\mu}_i \mathbf{A}_{11}) \mathbf{Z} \); 
   2.3.2 Solve (in place) \( \begin{pmatrix} \mathbf{E}_{11} + \mu_i \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathbf{Z} \\ 0 \end{pmatrix} \); 
   2.3.3 \( U_0 \leftarrow [U_0, \mathbf{Z}] \); 
   end 
   2.4 \( U \leftarrow \left[ U, \sqrt{2|\text{Re}(\mu_i)|} U_0 \right] \);
   \% Update and truncate SVD(\( U \)); 
   2.5 \( \mathbf{B} \leftarrow (\mathbf{E}_{11} - \bar{\mu}_i \mathbf{A}_{11}) \mathbf{Z} \).
end
Model Problem: Oseen Equations

\[
\begin{align*}
\frac{\partial}{\partial t} v(x, t) + (a(x) \cdot \nabla) v(x, t) & = \nu \Delta v(x, t) + \nabla p(x, t) \\
& = \chi_{\Omega_g}(x) g_{\Omega}(x, t) \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot v(x, t) & = 0 \quad \text{in } \Omega \times (0, T), \\
(-p(x, t) I + \nu \nabla v(x, t)) n(x) & = 0 \quad \text{on } \Gamma_n \times (0, T), \\
v(x, t) & = 0 \quad \text{on } \Gamma_d \times (0, T), \\
v(x, t) & = g_{\Gamma}(x, t) \quad \text{on } \Gamma_g \times (0, T), \\
v(x, 0) & = v_0(x) \quad \text{in } \Omega,
\end{align*}
\]
Channel Geometry and Grid

Figure: The channel geometry and coarse grid

EXAMPLE 1.

\[ y(t) = \int_{\Omega_{\text{obs}}} -\partial_{x_2} v_1(x, t) + \partial_{x_1} v_2(x, t) \, dx \]

over the subdomain \( \Omega_{\text{obs}} = (1, 3) \times (0, 1/2) \).
Model Reduction Results

<table>
<thead>
<tr>
<th>$n_v$</th>
<th>$n_p$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1352</td>
<td>205</td>
<td>13</td>
</tr>
<tr>
<td>5520</td>
<td>761</td>
<td>14</td>
</tr>
<tr>
<td>12504</td>
<td>1669</td>
<td>15</td>
</tr>
<tr>
<td>22304</td>
<td>2929</td>
<td>15</td>
</tr>
</tbody>
</table>

**Table:** Number $n_v$ of semidiscrete velocities $\mathbf{v}(t)$, number $n_p$ of semidiscrete pressures $\mathbf{p}(t)$, and size $k$ of the reduced order velocities $\hat{\mathbf{v}}(t)$ for various uniform refinements of the coarse grid.
Hankel S-vals and Convergence ADI

The largest Hankel singular values

Convergence of the multishift ADI Algorithm

Figure: The left plot shows the largest Hankel singular values and the threshold $\tau \sigma_1$. The right plot shows the normalized residuals $\|\tilde{B}_k\|_2$ generated by the multishift ADI Algorithm for the approximate solution of the controllability Lyapunov equation (○) and of the observability Lyapunov equation (★).
Figure: Time response (left) and frequency response (right) for the full order model (circles) and for the reduced order model (solid line).
Figure: Velocities generated with the full order model (left column) and with the reduced order model (right column) at $t = 1.0996, 2.9845, 3.7699, 4.86965, 6.2832$ (top to bottom).
Domain Decomposition Model Reduction

Systems with Local Nonlinearities

Linear Model in Large Domain $\Omega_1$
coupled to

Non-Linear Model in Small Domain $\Omega_2$

Linear Model provides B.C.’s to Non-Linear Model

with R. Glowinski
Simple 1-D Model Problem

\[ \Omega_1 \cup \Omega_2 \cup \Omega_3 \]

\[ \Omega_1 = (-10, -1), \quad \Omega_2 = (-1, 1), \quad \Omega_3 = (1, 10) \]

\[ \Omega_1 : \text{Convection-Diffusion} \]
\[ \Omega_2 : \text{Burgers Equation} \]
\[ \Omega_3 : \text{Convection-Diffusion} \]
Equations

\[ \rho_k \frac{\partial y_k}{\partial t}(x, t) - \mu_k \frac{\partial^2 y_k}{\partial x^2}(x, t) = S_k(x, t), \quad (x, t) \in \Omega_k \times (0, T), \]

\[ y_k(x, 0) = y_{k0}(x), \quad x \in \Omega_k, \; k = 1, 3, \]

\[ \frac{\partial y_1}{\partial x}(-10, t) = 0, \quad \frac{\partial y_3}{\partial x}(10, t) = 0 \quad t \in (0, T), \]

\[ \rho_2 \frac{\partial y_2}{\partial t}(x, t) - \mu_2 \frac{\partial^2 y_2}{\partial x^2} + y_2 \frac{\partial y_2}{\partial x}(x, t) = 0, \quad (x, t) \in \Omega_2 \times (0, T), \]

\[ y_2(x, 0) = y_{20}(x), \quad x \in \Omega_2, \]

with appropriate interface conditions
Semi-Discrete Equations

\[ M_1^{ll} \frac{d}{dt} y_1 + A_1^{ll} y_1 + M_1^{l\Gamma} \frac{d}{dt} y_1^{\Gamma} + A_1^{l\Gamma} y_1^{\Gamma} = B_1^{l} u_1, \]

\[ M_2^{ll} \frac{d}{dt} y_2 + A_2^{ll} y_2 + M_2^{l\Gamma} \frac{d}{dt} y_\Gamma + A_2^{l\Gamma} y_\Gamma + N^l (y_2, y_\Gamma) = 0, \]

\[ M_3^{ll} \frac{d}{dt} y_3 + A_3^{ll} y_3 + M_3^{l\Gamma} \frac{d}{dt} y_2^{\Gamma} + A_3^{l\Gamma} y_2^{\Gamma} = B_3^{l} u_3. \]
Equations to Reduce

Inputs to System 1: \( M_1^{} \Gamma d_t y_{12}^{\Gamma}, A_1^{} y_{12}^{\Gamma} \) and \( B_1^{} u_1 \)

Outputs of System 1: \( C_1^{} y_1^{\Gamma}, M_1^{} d_t y_1^{\Gamma} + A_1^{} y_1^{\Gamma} \)

Apply Model Reduction to

\[
M_1^{ll} \frac{d}{dt} y_1^{l} = -A_1^{ll} y_1^{l} - A_1^{} y_{12}^{\Gamma} + B_1^{} u_1
\]

\[
z_1^{l} = C_1^{} y_1^{l}, \quad z_1^{\Gamma} = A_1^{} y_1^{\Gamma}.
\]
**Dimension Reduction**

**Table:** Dimension of the full and of the reduced order models for various discretization parameters $N_1, N_2, N_3$ and $\tau = 10^{-4}$.

<table>
<thead>
<tr>
<th>$N_1 = N_3$</th>
<th>$N_2$</th>
<th>size of full model</th>
<th>size of ROM</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>201</td>
<td>41</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>401</td>
<td>63</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>801</td>
<td>107</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>381</td>
<td>43</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>761</td>
<td>67</td>
</tr>
</tbody>
</table>
**Time Response**

![Graph Showing Time Response](attachment:image.png)

**Figure:** Outputs 1, 2, 3 of the full order system corresponding to the discretization $N_1 = N_2 = N_3 = 10$ are given by dotted, dashed and solid lines, respectively. Outputs 1, 2, 3 of the reduced order system are given by *, ○ and □, respectively.

---

D.C. Sorensen
State Approximation

Figure: Solution of the reduced order discretized PDE (left) and error between the solution of the discretized PDE and the reduced order system (right) for discretization $N_1 = N_2 = N_3 = 10$. 
Balanced Truncation on a Compartmental Neuron Model

Steve Cox
Tony Kellems

Undergrads Nan Xiao and Derrick Roos

Quasi-active integrate and fire model.
Complex model dimension 6000
Faithfully approximated with 10-20 variable ROM
Neuron Cell

90 µm
Neuron Model

Full Non-Linear Model

\[ I_{j,\text{syn}} \] is the synaptic input into branch \( j \)

\[
\frac{a_j}{2R_i} \partial_{xx} v_j = C_m \partial_t v_j + G_{Na} m_j^3 h_j (v_j - E_{Na})
+ G_K n_j^4 (v_j - E_K) + G_l (v_j - E_l) + I_{j,\text{syn}}(x, t)
\]

Kinetics of the potassium (\( n \)) and sodium (\( h, m \)) channels

\[
\partial_t m_j = \alpha_m(v_j)(1 - m_j) - \beta_m(v_j)m_j \\
\partial_t h_j = \alpha_h(v_j)(1 - h_j) - \beta_h(v_j)h_j \\
\partial_t n_j = \alpha_n(v_j)(1 - n_j) - \beta_n(v_j)n_j.
\]
Linearized Neuron Model

Quasi-Active Approximation

\[
\frac{a_j}{2R_i} \partial_{xx} \tilde{v}_j = C_m \partial_t \tilde{v}_j + G_{Na}\{m^3 \tilde{h} \tilde{v}_j - (3 \tilde{m} \tilde{m}^2 \tilde{h} + \tilde{m}^3 \tilde{h}_j)E_{Na}\}
+ G_K\{n^4 \tilde{v}_j - 4 \tilde{n}_j n^3 E_K\} + G_l \tilde{v}_j + I_{j,\text{syn}}
\]

\[
\partial_t \tilde{m}_j = \sigma_m \tilde{v}_j - \tilde{m}_j / \tau_m
\]

\[
\partial_t \tilde{h}_j = \sigma_h \tilde{v}_j - \tilde{h}_j / \tau_h
\]

\[
\partial_t \tilde{n}_j = \sigma_n \tilde{v}_j - \tilde{n}_j / \tau_n
\]

where

\[
\tau_g \equiv 1/(\alpha_g(0) + \beta_g(0))
\]

\[
\sigma_g \equiv \alpha'_g(0)(1 - \bar{g}) - \beta'_g(0) \bar{g}, \quad g = m, h, n.
\]
Cell Response - Near Threshold

Morphology: RC−3−04−04−B.asc

Lin HH
BT
Full HH
Hankel Singular Value Decay

Morphology: RC−3−04−04−B.asc
Normalized Hankel Singular Values
Error

5 Stimulated Dendrites

Number of Singular Values used (100 runs/value)
Neural ROM Results

- Interesting example of many-input, single-output system
- Simulation time single cell 14 sec Full vs 0.01 sec Reduced
- Ultimate goal is to simulate a few-Million neuron system over a minute of brain-time
- Currently limited to a 10K neuron system over a few brain-seconds
- Parallel computing required
Summary

CAAM TR07-02, M. Heinkenschloss, D. C. S., & K. Sun
CAAM TR07-14, K. Sun, R. Glowinski, M. Heinkenschloss, DCS.

Gramian Based Model Reduction:  Balanced Reduction
Solving Large Lyapunov Equations:  Approximate Balancing
                           1M vars now possible
                           Parameter Free

Balanced Reduction of Oseen Eqns:  Extension to
                                 Descriptor System

Multi-Shift ADI Without Explicit Projectors:
Only need Saddle Point Solver  Sparse Direct or Iterative

Domain Decomposition - Systems with Local Nonlinearities

Neural Modeling - Single Cell ROM ⇒ Many Interactions